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Simple nuclear C^*-algebras of tracial topological rank one

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Abstract

We give a classification theorem for unital separable nuclear C^*-algebras with tracial rank no more than one. Let A and B be two unital separable simple nuclear C^*-algebras with $TR(A), TR(B) \leqslant 1$ which satisfy the universal coefficient theorem. We show that $A \cong B$ if and only if there is an order and unit preserving isomorphism

$$\gamma = (\gamma_0, \gamma_1, \gamma_2) : \big(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A)\big) \cong \big(K_0(B), K_0(B)_+, [1_B], K_1(B), T(B)\big),$$

where $\gamma_2^{-1}(\tau)(x) = \tau(\gamma_0(x))$ for each $x \in K_0(A)$ and $\tau \in T(B)$.

Keywords: TAI C^*-algebras; Classification of nuclear C^*-algebras; Tracial rank one

1. Introduction

This paper is a part of the program to classify separable nuclear C^*-algebras initiated by George A. Elliott (see [14] and [16]). By a classification theorem for a class of nuclear C^*-algebras, one means the following: two C^*-algebras in the class with the same K-theoretical data are isomorphic (as C^*-algebras) and the range of the invariant can be described for the class so that given a set of K-theoretical data in the range there is a C^*-algebra in the class

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which possesses the given K-theoretical data. By the K-theoretical data, one usually means the Elliott invariant which contains the K-theory and traces, at least for the simple case. In this paper we are only interested in simple C^*-algebras with lower rank. By C^*-algebras of lower rank, one often means that the C^*-algebras have real rank zero, or stable rank one. Many important C^*-algebras which arise naturally are of real rank zero or stable rank one. Notably, all purely infinite simple C^*-algebras have real rank zero and many C^*-algebras arising from dynamical systems are of stable rank one. One of the classical results of this kind states that all irrational rotation C^*-algebras are simple nuclear C^*-algebras with real rank zero and stable rank one (see [17] and [46]).

One may view (simple) C^*-algebras of real rank zero and stable rank one as some kind of generalization of AF-algebras. A more suitable generalization of AF-algebras has been demonstrated to be C^*-algebras with tracial topological rank zero. Simple C^*-algebras with tracial topological rank zero have real rank zero, stable rank one, with weakly unperforated K_0 and are quasidiagonal. All simple AH-algebras with slow dimension growth and with real rank zero have tracial topological rank zero. This shows that simple C^*-algebras with zero tracial rank could have rich K-theory. Simple AH-algebras with slow dimension growth and with real rank zero have been classified in [18] (together with [9,21] and [22]). A classification theorem for unital nuclear separable simple C^*-algebras with tracial topological rank zero which satisfy the UCT was given in [38] (see also [32,36] and [10] for earlier references). Simple C^*-algebras with tracial topological rank zero are also called TAF (tracially AF) C^*-algebras.

This paper studies C^*-algebras of tracial rank one. A standard example of a C^*-algebra with stable rank one is of course $M_k(C([0,1]))$ (which also has tracial rank one). A notion of tracially approximately interval C^*-algebras (TAI C^*-algebras) is introduced in this paper—see Definition 2.2 below. It turns out that simple TAI C^*-algebras are the same as simple C^*-algebras with tracial topological rank no more than one. Roughly speaking, TAI C^*-algebras are those C^*-algebras whose finite subsets can be approximated by C^*-subalgebras which are finite direct sums of finite-dimensional C^*-algebras and matrix algebras over $C([0,1])$ in "measure" or rather in trace. It is proved here that simple TAI C^*-algebras have stable rank one. From a result of G. Gong [23] we observe that all simple AH-algebras with very slow dimension growth are in fact TAI C^*-algebras. It is also shown here that simple TAI C^*-algebras are quasidiagonal, their ordered K_0-groups are weakly unperforated and satisfy the Riesz interpolation property, and these C^*-algebras also satisfy the Fundamental Comparison Property of Blackadar.

Elliott, Gong and Li in [20] (also [23]) give a complete classification (up to isomorphism) for simple AH-algebras with bounded dimension growth by their K-theoretical data (an important special case can be found in K. Thomsen's work [55]). G. Gong also has a proof [24] that simple AH-algebras with very slow dimension growth can be rewritten as simple AH-algebras with bounded dimension growth (the proof of this article also implies that—see 10.6).

These C^*-algebras are nuclear separable simple C^*-algebras of stable rank one. Their work is a significant advance in classifying finite simple C^*-algebras after the remarkable result of [18] which classifies simple AH-algebras of real rank zero (with slow dimension growth). Therefore, it is the time to classify nuclear simple separable finite C^*-algebras with real rank other than zero *without assuming* that they are inductive limits (AH-algebras are inductive limits of finite direct sums of some standard homogeneous C^*-algebras) of certain special building blocks. The main purpose of this paper is to present such a result.

Sections 1–6, Section 8 and most of Section 9 were written in 1998. Together with later sections, the original preprint has two parts. A preliminary report on the results in the two-part preprint was reported in EU Conference on Operator Algebra in Copenhagen in August 1998.

Since then a great deal of progress on the subject has been made. The present paper absorbs both parts of the original preprint and reflects the new developments. But it is significantly shorter than the original preprint. More importantly, the main result of the paper has been greatly improved and a technical condition in original preprint has been removed. The main result of the paper is the following. Let A and B be two unital separable nuclear simple C^*-algebras with $TR(A) \leqslant 1$, $TR(B) \leqslant 1$ and satisfying the UCT. Then $A \cong B$ if and only if they have the same Elliott invariant—see Theorem 10.10.

Consider two C^*-algebras A and B as above. As in [20], we will construct the following approximately commutative diagram:

$$\begin{tikzcd} B \arrow[r, "\mathrm{id}_B"] \arrow[d, "L_1"'] & B \arrow[d, "L_2"] \\ A \arrow[r, "\mathrm{id}_A"] \arrow[ur, "\Phi_1"] & A. \end{tikzcd}$$

We then apply an approximate intertwining argument of Elliott to obtain an isomorphism. It requires two types of results: an existence theorem and a uniqueness theorem. The existence theorem should state that if A and B have the same Elliott invariant, then there are unital $*$-homomorphisms from A to B and from B to A which induce the isomorphism at the level of the Elliott invariant. However, the existence theorem that we proved in the process only provides a sequence of completely positive maps from A to B (and from B to A) which are eventually multiplicative. In order to get the approximately commutative diagram, one also needs the uniqueness theorem: two such maps which induce the same (partial) map at the level of the Elliott invariant are approximately unitarily equivalent. In other words, we also need a uniqueness theorem which works for maps that are not necessary homomorphisms.

An important fact is the following classification of monomorphisms from $\bigoplus_{k=1}^n M_{r(k)}(C([0, 1]))$ to a simple TAI-algebra. For any unital C^*-algebra C, denote by $T(C)$ the tracial state space of C (it could be an empty set). Let $A = \bigoplus_{k=1}^n M_{r(k)}(C([0, 1]))$ and B be a unital simple TAI C^*-algebra. Suppose that $\phi_i : A \to B$ $(i = 1, 2)$ are two unital monomorphisms which induce the same map at the level of K_0 and satisfy

$$\tau \circ \phi_1(a) = \tau \circ \phi_2(a)$$

for all $a \in A$ and all $\tau \in T(B)$. Then there exists a sequence of unitaries $u_n \in B$ such that

$$\lim_{n\to\infty} u_n^* \phi_1(a) u_n = \phi_2(a) \quad \text{for all } a \in A.$$

Combining this with the more general uniqueness theorem of [35], we are able to obtain a uniqueness theorem for nuclear separable simple TAI C^*-algebras. As in [20], our invariant includes not only Banach algebra K-theory, but also an additional datum, namely, the tracial state space $T(A)$ together with a pairing of $T(A)$ and $K_0(A)$. Since traces are part of the invariant as in [15, 26] and [20], we also use some earlier results of J. Cuntz and G.K. Pedersen. However, we are able to avoid some difficult topological techniques involving higher-dimensional CW-complexes in [20]. We also show that the set of Elliott invariants for unital separable simple C^*-algebras with $TR(A) \leqslant 1$ which satisfy the UCT is the same as that of simple AH-algebras with no dimension growth as described by J. Villadsen [57] (see 10.2 below). The uniqueness theorem also has to be adjusted to deal with other complications caused by the fact that our C^*-algebras are no longer assumed to have real rank zero. A careful treatment on exponential length is needed. Our existence theorem also needs to be improved from that in [38]. The existence theorem should also control the exponential length. It turns out that when C^*-algebras are assumed to have only torsion K_1, the proof can be made much shorter. This is done without using de la Harpe and Skandalis determinants as in [20].

The paper is organized as follows. Section 2 gives the definition of TAI C^*-algebras. Section 3 gives some elementary properties of simple TAI C^*-algebras. In Section 4, we show that simple TAI C^*-algebras have stable rank one, weakly unperforated K_0, the fundamental comparison property, and are MF. Starting from Section 5, we will use term "simple C^*-algebras with $TR(A) \leqslant 1$" instead of "TAI C^*-algebra A." Even though that the term "$TR(A) \leqslant 1$" has appeared in [33], the term "TAI" has been used and results in the first 4 sections have been quoted in a number of places including [33]. We feel that we can keep the literature consistent by keeping the term "TAI" in the first 4 sections here. In Section 5, we show that every simple (nonelementary) C^*-algebra A with $TR(A) \leqslant 1$ is tracially approximately divisible. We also give a classification theorem for monomorphisms from $M_r(C([0,1]))$ to a unital simple TAI C^*-algebra mentioned earlier. In Section 6, we study the unitary group of a simple C^*-algebra A with $TR(A) \leqslant 1$. Exponential rank of a simple C^*-algebra A with $TR(A) \leqslant 1$ is proved to be no more than $3+\varepsilon$ in the sense of [44]. Let $CU(A)$ be the closure of commutator group of $U(A)$. We show that $U_0(A)/CU(A)$ is always divisible, and if A is simple and $TR(A) \leqslant 1$ then $U_0(A)/CU(A)$ is torsion free. In Section 7, we present some results concerning homomorphisms from $U(C)/CU(C)$ to $U(B)/CU(B)$, where B is a unital simple C^*-algebra with $TR(B) \leqslant 1$ and C is a very special unital C^*-algebra. These results may be viewed as part of the existence theorem which controls the exponential length of unitaries under certain maps. In Section 8, we present a uniqueness theorem suitable to be used in the proof of 10.4 which is based on results in [35]. One immediate consequence of it is the following. Let A be a unital separable simple nuclear C^*-algebra with $TR(A) \leqslant 1$ and with torsion $K_1(A)$. Then an automorphism $\alpha : A \to A$ is approximately inner if and only if $[\alpha] = [\mathrm{id}_A]$ in $KL(A)$ and $\tau \circ \alpha(x) = \tau(x)$ for each self-adjoint $x \in A$ and $\tau \in T(A)$. In Section 9, we present several existence theorems. The purpose is to establish a map from A to B if $TR(A) \leqslant 1$, $TR(B) \leqslant 1$ and $(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A)) = (K_0(B), K_0(B)_+, [1_B], K_1(B), T(B))$. Finally, in Section 10, we give the proof of the main theorem—Theorems 10.4 and 10.10.

The following terminology and notation will be used throughout this paper.

Let A be a C^*-algebra.

(i) Two projections in A are said to be equivalent if they are Murray–von Neumann equivalent. We write $p \preccurlyeq q$ if p is equivalent to a projection in qAq. We use $[p]$ for the equivalence class of projections equivalent to p. Let $a \in A_+$. We write $p \preccurlyeq a$ if $p \preccurlyeq q$ for some projection $q \in \overline{aAa}$.

(ii) An element in A is said to be *full* if the (closed) ideal generated by the element is A itself. Every nonzero element in a simple C^*-algebra is full.

(iii) Let $\varepsilon > 0$, $\mathcal{F}$ and S be a subset of A. We write $x \in_\varepsilon S$ if there exists $y \in S$ such that $\|x - y\| < \varepsilon$, and write $\mathcal{F} \subset_\varepsilon S$, if $x \in_\varepsilon S$ for all $x \in \mathcal{F}$.

(iv) Let A be a C^*-algebra. Denote by A_{sa} the set of all self-adjoint elements of A and denote by A_+ the set of all positive elements of A.

(v) Let $\mathcal{G} \subset A$ and $\delta > 0$. A contractive completely positive linear map $L : A \to B$ is said to be $\mathcal{G}$-δ-multiplicative if

$$\big\|L(ab) - L(a)L(b)\big\| < \delta \quad \text{for all } a, b \in \mathcal{G}.$$

(vi) Let X be a compact metric space and $h : PM_r(C(X))P \to A$, where $P \in M_r(C(X))$ is a projection, be a homomorphism. We say h is *homotopically trivial*, if h is homotopic to a point-evaluation. A contractive completely positive linear map $L : PM_r(C(X))P \to A$ is said to be *homotopically trivial*, if L factors through a homotopically trivial homomorphism, i.e., $L = L' \circ h$, where h is homotopically trivial.

2. Definition of tracially AI C^*-algebras

2.1. Definition. We denote by $\mathcal{I}$ the class of all unital C^*-algebras with the form $\bigoplus_{i=1}^n B_i$, where each $B_i \cong M_{k(i)}$ for some integer $k(i)$ or $B_i \cong M_{k(i)}(C([0,1]))$. Let $A \in \mathcal{I}$. We have the following well-known facts.

(i) Every C^*-algebra in $\mathcal{I}$ is of stable rank one.
(ii) Two projections p and q in a C^*-algebra $A \in \mathcal{I}$ are equivalent if and only if $\tau(p) = \tau(q)$ for all $\tau \in T(A)$.
(iii) For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: if $L : A \to B$ is a $\mathcal{G}$-δ-multiplicative contractive completely positive linear map, where B is a C^*-algebra, then there exists a homomorphism $h : A \to B$ such that

$$\big\|h(a) - L(a)\big\| < \varepsilon \quad \text{for all } a \in \mathcal{F}.$$

These facts will be used throughout the paper without further notice.

2.2. Definition. A unital C^*-algebra A is said to be *tracially AI* (TAI) if for any finite subset $\mathcal{F} \subset A$ containing a nonzero element b, $\varepsilon > 0$, integer $n > 0$ and any full element $a \in A_+$, there exist a nonzero projection $p \in A$ and a C^*-subalgebra $I \subset A$ with $I \in \mathcal{I}$ and $1_I = p$, such that:

(1) $\|[x, p]\| < \varepsilon$ for all $x \in \mathcal{F}$,
(2) $pxp \in_\varepsilon I$ for all $x \in \mathcal{F}$ and $\|pbp\| \geqslant \|b\| - \varepsilon$,
(3) $n[1-p] \leqslant [p]$ and $1 - p \precsim a$.

A non-unital C^*-algebra A is said to be TAI if $\tilde{A}$ is TAI.

In 4.10, we show that, if A is simple, condition (3) can be replaced by

(3′) $1 - p$ is unitarily equivalent to a projection in eAe for any previously given nonzero projection $e \in A$.

If A has the Fundamental Comparability (see [2]), condition (3) can be replaced by

(3″) $\tau(1-p) < \sigma$ for any prescribed $\sigma > 0$ and for all normalized quasi-traces of A.

From the definition, one sees that the part of A which may not be approximated by C^*-algebras in $\mathcal{I}$ has small "measure" or trace. Note in the above, if $\mathcal{I}$ is replaced by finite-dimensional C^*-algebras, then it is precisely the definition of TAF C^*-algebras (see [31]).

2.3. Example. Every AF-algebra is TAI. Every TAF C^*-algebra introduced in [31] is a TAI C^*-algebra. However, in general, TAI C^*-algebras have real rank other than zero. In 4.5 we will show that every simple TAI C^*-algebra has stable rank one, which implies that simple TAI C^*-algebras have real rank one or zero. It is obvious that every direct limit of C^*-algebras in $\mathcal{I}$ is a TAI C^*-algebra. These C^*-algebras provide many examples of TAI C^*-algebras that have real rank one. However, TAI C^*-algebras may not be inductive limits of C^*-algebras in $\mathcal{I}$.

Let $A = \lim_{n\to\infty}(A_n, \phi_{n,m})$, where $A_n = \bigoplus_{i=1}^{s(n)} P_{n,i} M_{(n,i)}(C(X_{n,i}))P_{n,i}$, $X_{n,i}$ is a finite-dimensional compact metric space and $P_{n,i} \in M_{(n,i)}(C(X_{n,i}))$ is a projection for all n and i. Such a C^*-algebra is called an AH-algebra. Suppose that A is unital. Following [23], A is said to have *very slow dimension growth* if

$$\lim_{n\to\infty} \min_i \frac{\operatorname{rank}(P_{n,i})}{(\dim X_{n,i}+1)^3} = \infty.$$

A is said to have *no dimension growth* if there is an integer $m > 0$ such that $\dim X_{n,i} \leqslant m$. Note these C^*-algebras may not be of real rank zero. Since these C^*-algebras could have non-trivial K_1-groups (see 10.1), they are not inductive limits of C^*-algebras in $\mathcal{I}$. In [31], example of simple TAF C^*-algebras which are non-nuclear was given. In particular, there are simple TAI C^*-algebras that are not even nuclear.

2.4. Lemma. *Let a be a positive element in a unital C^*-algebra A with* $\operatorname{sp}(a) \subset [0,1]$. *Then for any $\varepsilon > 0$, there exists $b \in A_+$ such that* $\operatorname{sp}(b)$ *is a union of finitely many mutually disjoint closed intervals and finitely many points and*

$$\|a-b\| < \varepsilon.$$

Proof. Fix $\varepsilon > 0$. Let $I_1, I_2, \ldots, I_k$ be all disjoint closed intervals in $\operatorname{sp}(a)$ with length at least $\varepsilon/8$ such that if $I \supset I_j$ is an interval, then $I \not\subset \operatorname{sp}(a)$. Let $d' = \min\{\operatorname{dist}(I_i, I_j), i \neq j\}$ and $d = \min(d'/2, \varepsilon/16)$.

Choose $J_i = \{\xi \in [0,1]\colon \operatorname{dist}(\xi, I_j) < d_i\}$, $i = 1, 2, \ldots, k$ with $d_i \leqslant d$ and the endpoints of J_i are not in $\operatorname{sp}(a)$. Since the endpoints of J_i are not in $\operatorname{sp}(a)$, there are open intervals $J_i' \subset J_i$ such that $\bar{J}_i' \subset J_i$ and $I_i \subset J_i'$. Set $Y = \operatorname{sp}(a) \setminus (\bigcup_{i=1}^k J_i)$. Then $Y = \operatorname{sp}(a) \setminus (\bigcup_{i=1}^k J_i') = \operatorname{sp}(a) \setminus (\bigcup_{i=1}^k \bar{J}_i')$. Since Y is compact and Y contains no intervals with length more than $\varepsilon/8$, it is routine to show that there are finitely many disjoint closed intervals $K_1, K_2, \ldots, K_n$ in $[0,1] \setminus (\bigcup_{i=1}^k J_i)$ with length no more than $9\varepsilon/64$ such that $Y \subset \bigcup_{j=1}^m K_j$. Note that $\{\bar{J}_1', \bar{J}_2', \ldots, \bar{J}_k', K_1, K_2, \ldots, K_n\}$ are disjoint closed intervals. Fix a point $\xi_j \in K_j$, $j = 1, 2, \ldots, m$. One can define a continuous function $f : (\bigcup_{i=1}^k \bar{J}_i') \cup (\bigcup_{s=1}^n K_s) \to [0,1]$ which maps each J_i onto I_i, $i = 1, 2, \ldots, k$ and maps K_j to a single point ξ_j such that

$$\left|f(\xi) - \xi\right| < \varepsilon/2 \quad \text{for all } \xi \in [0,1].$$

Define $b = f(a)$. We see that b meets the requirements of the lemma. $\square$

2.5. Theorem. *Let A be a unital simple AH-algebra with very slow dimension growth. Then A is TAI.*

Proof. By 1.3.3, 1.3.4 and 4.23 of [23] (and [19]), to show that A is TAI, it suffices to assume that $A = \lim_{n\to\infty}(A_n, \phi_{n,m})$, where $A_n = \bigoplus_{i=1}^{i(n)} M_{n(i)}(C(X_{n,i}))$, $X_{n,i}$ are simplicial complexes and $\phi_{n,m}$ are injective (see also 3.4 below). Moreover, we may also assume that A satisfies the condition of very slow dimension growth.

Let $\varepsilon > 0$, $\mathcal{F} \subset A$ be a finite subset and $e \in A$ be a non-zero projection. To verify (1), (2) and (3′) in 2.2, without loss of generality, we may assume that $\mathcal{F} \subset A_1$ and $e \in A_1$. By considering each summand separately, without loss of generality, we may also assume that $A_1 = M_r(C(X))$ for some finite simplicial complex and integer $r \geqslant 1$. Let $\mathcal{F}_1 \subset C(X)$ be a finite subset such that $\mathcal{F} \subset \{(f_{i,j})_{r\times r}\colon f_{i,j} \in \mathcal{F}_1\}$.

Let $J > r+1$ be an integer. Let $\varepsilon/2r^2 > \eta > 0$ such that $|f(x) - f(x')| < \varepsilon/9r^2$ for all $f \in \mathcal{F}_1$ whenever $\mathrm{dist}(x, x') < 2\eta$. Let $\delta > 0$ and L be as in Theorem 4.35 corresponding to $\varepsilon/2r^2$, η and $\mathcal{F}_1$ above. Since A is simple, as in 4.36 of [23], each partial map of $\phi_{1,m}$ (for sufficiently large m) has the property $\mathrm{sdp}(\eta/32, \varepsilon/2r^2)$. To simplify notation, without loss of generality, we may assume that $A_m = M_k(C(Y))$ and $\mathrm{rank}(\phi_{1,m}(1)) > 2JL^2 2^L(\dim X + \dim Y + 1)^3$, where Y is a finite simplicial complex. To simplify notation, by considering each summand separately, without loss of generality, we may assume that Y is connected. Since A is simple, by choosing a larger m, we further assume that $e \in M_k(C(Y))$ is a non-zero projection which has the rank at least $\mathrm{rank}(\phi_{1,m}(1))/r$.

By applying 4.35 of [23], there are three mutually orthogonal projections $Q_0, Q_1, Q_2 \in A_m$ and homomorphisms $\psi_i : A_1 \to Q_i A_m Q_i$ $(i = 0, 1, 2)$ such that:

(1) $\phi_{1,m} = Q_0 + Q_1 + Q_3$;
(2) $\|\phi_{n,m}(f) - (\phi_0(f) \oplus \phi_1(f) \oplus \phi_2(f))\| < \varepsilon/2$ for all $f \in \mathcal{F}$;
(3) ψ_2 factors through $M_r(C([0, 1])$;
(4) ψ_1 has finite-dimensional range;
(5) $J[Q_0] \leqslant [Q_1]$.

Put $\psi = \psi_1 \oplus \psi_2$. It follows from Lemma 2.4 that there is a unital C^*-subalgebra $B_1 \in \mathcal{I}$ of $(Q_1 + Q_2)A_m(Q_1 + Q_2)$ such that

$$\psi(f) \in_\varepsilon B_1 \quad \text{for } f \in \mathcal{F}.$$

We also have

$$[Q_0] \leqslant [e].$$

Thus, A is of TAI. □

3. Elementary properties of simple TAI C^*-algebras

3.1. Lemma. *For any $d > 0$ there are $f_1, f_2, \ldots, f_m \in C([0, 1])_+$ with the following properties. For any n, and any positive element $x \in B = M_n(C([0, 1]))$ with $\|x\| \leqslant 1$ if there exist $a_{ij} \in B$, $i = 1, 2, \ldots, n(j)$, $j = 1, 2, \ldots, m$ with*

$$\left\| \sum_{i=1}^{n(j)} a_{i,j} f_j(x) a_{i,j}^* - 1_A \right\| < 1/2, \quad j = 1, 2, \ldots, m,$$

then, for any subinterval J of $[0,1]$ with $\mu(J) \geqslant d$ (μ is the Lebesgue measure), $\mathrm{sp}(\pi_t(x)) \cap J \neq \emptyset$ for all $t \in [0,1]$, where $\pi_t : B \to M_n$ is the point evaluation at t. Moreover, denote by $N = \max\{n(j)\colon\ j = 1, 2, \ldots, m\}$,

$$\left|\mathrm{sp}\bigl(\pi_t(x)\bigr) \cap J\right| \geqslant 1/N \left|\mathrm{sp}\bigl(\pi_t(x)\bigr)\right|,$$

where $|S|$ means the number of elements in the finite set S (counting multiplicities).

Proof. Divide $[0,1]$ into m closed subintervals $\{J_j\}$ each of which has the same length $< d/4$. Let $f_j \in C([0,1])$ be such that $0 \leqslant f_j \leqslant 1$, $f_j(t) = 1$ for $t \in J_j$ and $f_j(t) = 0$ for $\mathrm{dist}(t, J_j) \geqslant \mu(J_j)$. Note that, for any subinterval J with $\mu(J) \geqslant d$, there exists j such that $J_j \subset J$. For any $t \in [0,1]$, set

$$I = \bigl\{g \in B\colon\ g(t) = 0\bigr\}.$$

Then I is a (closed) ideal of A. If $\mathrm{sp}(\pi_t(x)) \cap J = \emptyset$, there would be j such that $\pi_t(f_j(x)) = 0$. Therefore $f_j \in I$. But this is impossible, since there is an element $z \in B$ with

$$z\left(\sum_{i=1} a_{i,j} f_j(x) a_{i,j}^*\right) = 1_B.$$

For the last part of the lemma, fix $t \in [0,1]$ and an interval J with $t \in J$ and $\mu(J) \geqslant d$. Let $\pi_t(B) = M_{l(t)}$. Then $|\mathrm{sp}(\pi_t(x))| = l(t)$. Suppose that $J_j \subset J$ so that $f_j(t) = 0$ for all $t \notin J$. Let q_t be the spectral projection of $\pi_t(x)$ in $M_{l(t)}$ corresponding to J. Then $q_t \geqslant f_j(\pi_t(x))$. An elementary linear algebra argument shows that $\mathrm{rank}\, q_t \geqslant (1/N) l(i)$. □

3.2. Theorem. *Every unital simple C^*-algebra satisfying* (1) *and* (2) *in* 2.2 *has property* (SP)*, i.e., every hereditary C^*-subalgebra contains a nonzero projection.*

Proof. Let A be a unital simple C^*-algebra satisfying (1) and (2) and $B \subset A$ be a hereditary C^*-subalgebra. We may assume that A is not elementary. Thus B is not elementary. By p. 61 (item 4) in [1], there is $a \in B_+$ such that $\mathrm{sp}(a) = [0,1]$. It suffices to show that $B_1 = \overline{aAa}$ has a nonzero projection.

Let d be a positive number with $0 < d < 1/16$. Let $f_1, f_2, \ldots, f_m$ be as in 3.1. Since A is simple, there are $a_{ij} \in A$ such that

$$\left\|\sum_{i=1} a_{i,j} f_j(a) a_{i,j}^* - 1_A\right\| < 1/8, \quad j = 1, 2, \ldots, m.$$

Let $g \in C_0((0,1))$ with $0 \leqslant g \leqslant 1$, $g(t) = 1$ if $g \in [1/4, 3/4]$. Denote by $\mathcal{F}$ the subset

$$\bigl\{a, g(a), f_j(a), a_{i,j}, a_{ij}^*\colon\ i, j\bigr\}.$$

For any $\varepsilon > 0$, there exists a projection $p \in A$ and a C^*-subalgebra $C \in \mathcal{I}$ with $1_I = p$ such that

(1) $\|[b, p]\| < \varepsilon$ for all $b \in \mathcal{F}$,
(2) $pbp \in_\varepsilon C$ and $\|pap\| \geqslant 1 - \varepsilon$.

A standard perturbation argument shows that, for any $\eta > 0$, with sufficiently small ε, there is a homomorphism $\phi : C^*(pap) \to C$ (where $C^*(pap)$ is the C^*-subalgebra generated by pap) and there are $b_{ij} \in C$ such that

$$\|pap - \phi(pap)\| < \eta, \qquad \|\phi(f_j(a)) - pf_j(a)p\| < \eta,$$
$$\|\phi(g(a)) - pg(a)p\| < \eta, \quad \text{and} \quad \Big\|\sum_{i=1} b_{ij}\phi(f_j(pap))b_{ij}^* - p\Big\| < 1/4$$

for $j = 1, 2, \ldots, m$.

Write $C = \bigoplus_k M_{m(k)}(C([0,1]))$ and $C_k = M_{m(k)}(C([0,1]))$. By Lemma 3.1, with sufficiently small d, we may assume that

$$\big|\mathrm{sp}\big(\pi_t(\phi_k(pap))\big) \cap (0,1]\big| \geqslant 4$$

for each t and k, where π_t is the point-evaluation at $t \in [0,1]$, ϕ_k is the map from $C([0,1])$ to $M_{m(k)}(C([0,1]))$ induced by ϕ and where "$|\mathrm{sp}(\pi_t(\phi_k(pap)))|$" is the number of eigenvalues of $\pi_t(\phi_k(pap))$ counting multiplicities.

Fix k. For $t \in [0,1]$, let $J_t \subset [1/4, 3/4]$ be an open interval with $\mu(J_t) \geqslant d$ whose endpoints are not eigenvalues of $\pi_t(\phi_k(pap))$. Let V_t be an open neighborhood of t such that the end points of J_t are not eigenvalues of $\pi_y(\phi_k(pap))$ for $y \in V_t$. Let ξ_{J_t} be the characteristic function on J_t. Then using the continuous functional calculus we can define a continuous projection valued function $q_t : V_t \to M_{m(k)}$ by $q_t(y) = \xi_{J_t}(\phi_k(pap))$. From the previous paragraph, $\mathrm{rank}(q_t(y)) \geqslant 4$. It follows from Proposition 3.2 in [13] that there exists a nonzero projection $q \in M_{m(k)}(C([0,1]))$ such that $q(t) \leqslant f(\phi(pap))(t)$ for all $t \in [0,1]$, where $f \in C_0((0,1))$ with $0 \leqslant f \leqslant 1$, $f(t) = 1$ if $t \in [1/4, 3/4]$ (see also the proof of 1.4 of [43]). In particular, $qf(\phi(pap)) = q$.

Let $b = f(a)$. We estimate that

$$\|bq - q\| < 2\eta.$$

It is standard that if $\eta < 1/8$, there is a projection $q' \subset \overline{bAb}$ such that

$$\|q' - q\| < 1/2.$$

This implies that $\overline{bAb} \subset \overline{aAa} \subset B$ contains a nonzero projection (q'). $\square$

3.3. Corollary. *Let A be a unital simple C^*-algebra satisfying* (1) *and* (2) *in* 2.2. *Then, for any integer N, we may assume that $I = \bigoplus_{i=1}^k M_{m_i}(C([0,1])) \bigoplus_{j=1}^l M_{n_i}$ where $m_i, n_i \geqslant N$.*

Proof. In the proof of 3.2, we see that if $1/2d \geqslant N$, since $\mathrm{sp}(\pi_t(pap)) \cap J_j \neq \emptyset$ for each j, then $\pi_t(pap)$ has at least N distinct eigenvalues (see also the proof of 3.1). Therefore, each summand C in the proof 3.2 has rank at least N. $\square$

3.4. Proposition. *Let A be a unital TAI C^*-algebra and $e \in A$ be a full projection. Then eAe satisfies* (1) *and* (2) *in* 2.2, *and for any full positive element $a \in eAe$, we can have*

(3′) $1 - p \precsim a$.

If A is also simple, eAe is TAI.

Proof. Fix $\varepsilon > 0$, a finite subset $\mathcal{F} \subset eAe$, an integer $n > 0$ and nonzero elements $a, b \in eAe$ with $a \geqslant 0$ and $b \in \mathcal{F}$. Let $\mathcal{F}_1 = \{e\} \cup \mathcal{F}$. Since A is TAI, there exists $q \in A$ and a C^*-subalgebra $C \in \mathcal{I}$ with $1_C = q$ such that:

(i) $\|[x, q]\| < \varepsilon/64$ for all $x \in \mathcal{F}$,
(ii) $qxq \in_{\varepsilon/64} C$ for all $x \in \mathcal{F}$, and $\|qbq\| \geqslant \|b\| - \varepsilon/64$; and,
(iii) $n[1-q] \leqslant [q]$ and $1 - q \precsim a$.

Note that, by the second part of (ii), $qeq \neq 0$. We estimate that

$$\left\|(eqe)^2 - eqe\right\| < \varepsilon/64 \quad \text{and} \quad \|eqe - qeq\| < \varepsilon/32.$$

Therefore there is a projection $p \in eAe$ such that

$$\|p - eqe\| < \varepsilon/16.$$

Consequently, there is a projection $d \in C$ such that

$$\|d - p\| < \varepsilon/8.$$

Note that

$$\|qp - pq\| < \varepsilon/8 + \|qeqe - eqeq\| < \varepsilon/8 + \varepsilon/32 = 5\varepsilon/32,$$

and $B = dCd \in \mathcal{I}$. With $\varepsilon/2 < 1/2$, we obtain a unitary $u \in A$ such that

$$\|u - 1\| < \varepsilon/4 \quad \text{and} \quad u^* du = p.$$

Set $C_1 = u^* Bu$. Then $C_1 \in I$ and $C_1 \subset eAe$. Now $1_{C_1} = p$,

(1) $\|[x, p]\| < \varepsilon/2$ for all $x \in \mathcal{F}$,
(2) $pxp \in_{\varepsilon/2} C_1$ for all $x \in \mathcal{F}$ and $\|pbp\| \geqslant \|b\| - \varepsilon/2$.

We also have

$$\begin{aligned}
&\left\|(e-p) - (1-q)(e-p)(1-q)\right\| \\
&\quad \leqslant \left\|(e-p) - (e-p)(1-q) + q(e-p)(1-q)\right\| \\
&\quad < \left\|(e-p)q\right\| + \varepsilon/16 < 5\varepsilon/32 + \varepsilon/64 + \|qeq - qpq\| + \varepsilon/16 \\
&\quad < 5\varepsilon/32 + \varepsilon/64 + \varepsilon/16 + \|qeq - qeqeq\| + \varepsilon/16 < 9\varepsilon/32.
\end{aligned}$$

We have (with $\varepsilon < 1$)

(3′) $(e - p) \precsim (1 - q) \precsim a$.

Finally, if we assume that A is simple, by 3.2 and 3.3 there is a nonzero projection $p_1 \leqslant p$ such that $n[p_1] \leqslant [p]$. There is a nonzero projection $p_1 \in \overline{aAa}$. By applying 3.3, we obtain a nonzero projection $q_1 \leqslant p_1$ such that $n[q_1] \leqslant [p_1]$. Applying the first part of the proof to $(e-p)\mathcal{F}(e-p)$, we obtain a projection $p' \leqslant (e-p)$ and a unital C^*-subalgebra $C_2 \in \mathcal{I}$ with $1_{C_2} = p'$ such that:

(1″) $\|[(e-p)x(e-p), p']\| < \varepsilon/2$ for all $x \in \mathcal{F}$,
(2″) $p'xp' \in_{\varepsilon/2} C_2$ for all $x \in \mathcal{F}$, and
(3″) $(e-p-p') \precsim p_1$.

Now since $n[p_1] \leqslant [p] \leqslant [p+p']$ and $(e-p-p') \precsim (e-p) \precsim a$, we obtain

(3) $n[(e-p-p')] \leqslant [p+p']$ and $(e-p-p') \precsim a$.

We also have $\|[x, (p+p')]\| < \varepsilon$ and $(p+p')x(p+p') \in_\varepsilon C_1 \oplus C_2$ for all $x \in \mathcal{F}$. Hence eAe is TAI. □

3.5. Corollary. *If A is a unital simple TAI C^*-algebra, then condition* (2) *can be strengthened to*

(2′) *$pxp \in_\varepsilon B$ and $\|pxp\| \geqslant \|x\| - \varepsilon$ for all $x \in \mathcal{F}$.*

We omit the proof.

3.6. Theorem. *Let A be a unital simple C^*-algebra. Then A is TAI if and only if $M_n(A)$ is TAI for all n* (*or for some $n > 0$*).

Proof. If $M_n(A)$ is TAI, then by identifying A with a unital hereditary C^*-subalgebra of $M_n(A)$ and by using 3.4, we know A is TAI. It remains to prove the "only if" part.

We prove this in two steps. The first step is to prove that $M_n(A)$ satisfies (1) and (2) in 2.2. To do this, we let $\varepsilon > 0$ and $\mathcal{F}$ be a finite subset of the unit ball of $M_n(A)$. Set $\mathcal{G} = \{f_{ij} \in A\colon (f_{ij})_{n\times n} \in \mathcal{F}\}$. Note that $\mathcal{G} \subset A$. Since A is TAI, there exists a projection $p \in A$ and a unital C^*-subalgebra $B \in \mathcal{I}$ such that:

(1) $\|[x, p]\| < \varepsilon/2n^2$,
(2) $pxp \in_{\varepsilon/2n^2} B$ for all $x \in \mathcal{G}$ and for some $x_1 \in \mathcal{G}$, $\|px_1p\| \geqslant \|x_1\| - \varepsilon/2n^2$.

Put $P = \mathrm{diag}(p, p, \ldots, p) \in M_n(A)$ and $D = M_n(B)$. Then, it is easy to check that

(i) $\|[f, P]\| < \varepsilon$ and
(ii) $PfP \in_\varepsilon D$ for all $f \in \mathcal{F}$ and $\|pf_1p\| \geqslant \|f_1\| - \varepsilon$ (if f_1 is prescribed).

This completes the first step. Now we also know by 3.2 that $M_n(A)$ has (SP). Let $a \in M_n(A)$ be given. Choose any nonzero projection $e \in \overline{aM_n(A)a}$. Since $M_n(A)$ is simple and has (SP), by 3.1 in [31], there is a nonzero projection $q \leqslant e$ and $[q] \leqslant [1_A]$. Applying [31, 3.2], there exists a nonzero projection $q_1 \leqslant q$ such that $(n+1)[q_1] \leqslant [q]$. In the first step, we can also require, for any integer $N > 0$, that

(3) $N[1_A - p] \leqslant [p]$ and $1_A - p \precsim q_1$.

This implies that

(iii) $N[1_{M_n(A)} - P] \leqslant [P]$ and $(1_{M_n(A)} - P) \precsim q \precsim e$.

Therefore $M_n(A)$ is TAI. □

Next we show that every simple TAI C^*-algebra has the property introduced by Popa [45].

3.7. Proposition. *Let A be a unital simple TAI C^*-algebra. Then for any finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$, there exists a projection $p \in A$ and a finite-dimensional C^*-algebra $F \subset A$ with $1_F = p$ such that*

(P1) *$\|[x, p]\| < \varepsilon$ and*
(P2) *$pxp \in_\varepsilon F$ for all $x \in \mathcal{F}$ and $\|pxp\| \geqslant \|x\| - \varepsilon$ for all $x \in \mathcal{F}$.*

Proof. By 3.5 it is clear that it suffices to prove the following claim: for any unital C^*-subalgebra $B \in \mathcal{I}$, the proposition holds for any finite subset $\mathcal{F} \subset B \subset A$.

This can be further reduced to the case that $B = C([0,1]) \otimes M_k$. Moreover, it suffices to prove the claim for the case in which $B = C([0,1])$. In this reduced case, we only need to consider the case in which $\mathcal{F}$ contains a single element $x \in B$, where x is the identity function on $[0,1]$. Now, for any $\varepsilon > 0$, let $\xi_1, \xi_2, \ldots, \xi_n$ be $\varepsilon/4$-dense in $[0,1]$ and $\mathrm{dist}(\xi_i, \xi_j) > \varepsilon/8$ if $i \neq j$. Denote by f_i continuous functions with $0 \leqslant f_i \leqslant 1$, which are one on $(\xi_i - \varepsilon/32, \xi_i + \varepsilon/32)$ and zero on $[0,1] \setminus (\xi_i - \varepsilon/16, \xi_i + \varepsilon/16)$. By 3.2, there is a nonzero projection $e_i \in \overline{f_i A f_i}$. Note that $e_i e_j = 0$ if $i \neq j$. Set $p = \sum_{i=1}^n e_i$. We estimate that (see [29, Lemma 2])

$$\left\| x - \left[(1-p)x(1-p) + \sum_{i=1}^n \xi_i e_i \right] \right\| < \varepsilon/2 \quad \text{and}$$

$\|[p, x]\| < \varepsilon$ by (P1).

Let F_1 be the finite-dimensional C^*-subalgebra generated by $e_1, e_2, \ldots, e_n$. Then by (P2), $pxp \in_\varepsilon F_1$ and $\|pxp\| \geqslant \|x\| - \varepsilon$. □

4. The structure of simple TAI C^*-algebras

4.1. Theorem. *Every unital separable simple TAI C^*-algebra is MF* [4].

Proof. Let A be such a C^*-algebra and let $\{x_n\}$ be a dense sequence in the unit ball of A. By 3.7, there are projections $p_n \in A$ and finite-dimensional C^*-subalgebras B_n with $1_{B_n} = p_n$ such that

(1) $\|[p_n, x_i]\| < 1/n$, and
(2) $px_i p \in_{1/n} B_n$ and $\|p_n x_i p_n\| \geqslant \|x_i\| - 1/n$ for $i = 1, 2, \ldots, n$.

Let $\mathrm{id}_n : B_n \to B_n$ be the identity map and let $j : B_n \to M_{K(n)}$ be a unital embedding. We note that such j exists provided that $K(n)$ is large enough. By [41, 5.2], there exists a completely positive map $L_n' : p_n A p_n \to M_{K(n)}$ such that $L_n'|_{B_n} = j \circ \mathrm{id}_n$. Since L_n' is unital, by [41, 5.9 and

5.10], L_n' is a contraction. We define $L_n : A \to M_{K(n)}$ by $L_n(a) = L_n'(p_n a p_n)$. Let $y_{i,n} \in B_n$ such that $\|p_n x_i p_n - y_{i,n}\| < 1/n$, $n = 1, 2, \ldots$. Then

$$\left\|L_n(x_i) - p_n x_i p_n\right\| \leqslant \left\|L_n(x_i - y_{i,n}) - (y_{i,n} - p_n x_i p_n)\right\| < 2/n \to 0$$

as $n \to \infty$. Combining this with (1) above, we see that

$$\left\|L_n(ab) - L_n(a)L_n(b)\right\| \to 0$$

as $n \to \infty$. Define $\Phi : A \to \prod_{n=1}^{\infty} M_{m(n)}$ by sending a to $\{L_n(a)\}$. Then Φ is a completely positive map. Denote by $\pi : \prod_{n=1}^{\infty} M_{m(n)} \to \prod_{n=1}^{\infty} M_{m(n)} / \bigoplus_{n=1}^{\infty} M_{m(n)}$. Then

$$\pi \circ \Phi : A \to \prod_{n=1}^{\infty} M_{m(n)} / \bigoplus_{n=1}^{\infty} M_{m(n)}$$

is a (nonzero) homomorphism. Since A is simple, $\pi \circ \Phi$ is injective. It follows from [4, 3.22] that A is an MF-algebra. □

4.2. Corollary. *Every separable unital C^*-algebra satisfying* (P1) *and* (P2) *is MF.*

Proof. We actually proved this above. Note, simplicity is not needed for injectivity since $\|p_n x p_n\| \to \|x\|$. □

4.3. Proposition. *Every nuclear separable simple TAI C^*-algebra is quasidiagonal.*

Proof. As in [4], a separable nuclear MF C^*-algebra is NF, and it is quasidiagonal. In fact it is strong NF (see [5]). □

4.4. Corollary. *Every unital separable simple TAI C^*-algebra has at least one tracial state.*

Proof. It is well known that $\prod_{n=1}^{\infty} M_{m(n)} / \bigoplus_{n=1}^{\infty} M_{m(n)}$ has tracial states. Tracial states are defined by weak limits of tracial states on each $M_{m(n)}$. Let τ be such a tracial state. Then, in the proof of 4.2, let $t(a) = \tau \circ \pi \circ \Phi(a)$. □

4.5. Theorem. *A unital simple TAI C^*-algebra has stable rank one.*

Proof. Let A be a unital simple C^*-algebra. Take a nonzero element $a \in A$. We will show that a is a norm limit of invertible elements in A. So we may assume that a is not invertible and $\|a\| = 1$. Since A is finite, a is not one-sided invertible. For any $\varepsilon > 0$, by [50, 3.2], there is a zero divisor $b \in A$ such that $\|a - b\| < \varepsilon/2$. We further assume that $\|b\| \leqslant 1$. Therefore, by [50], there is a unitary $u \in A$ such that ub is orthogonal to a nonzero positive element $c \in A$. Set $d = ub$. Since A has (SP) (by 3.2), there exists a nonzero projection $e \in A$ such that $de = ed = 0$. Since A is simple and has (SP) (by 3.2), we may write $e = e_1 \oplus e_2$ with $e_2 \lesssim e_1$. Note that $d \leqslant (1 - e) \leqslant (1 - e_1)$. Moreover, $(1 - e_1)A(1 - e_1)$ is TAI.

Let $\eta > 0$ be a positive number. There is a projection $p \in (1 - e_1)A(1 - e_1)$ and a unital C^*-subalgebra $B \in \mathcal{I}$ with $1_B = p$ such that:

(1) $\|[x, p]\| < \eta$,
(2) $pxp \in_\eta B$ for all $x \in \mathcal{F}$, and
(3) $[1 - e_1 - p] \leqslant [e_2]$,

where $\mathcal{F}$ contains d. Thus, with sufficiently small η, we may assume that

$$\|d - (d_1 + d_2)\| < \varepsilon/16,$$

where $d_1 \in B$ and $d_2 \in (1 - e_1 - p)A(1 - e_1 - p)$.

Since C^*-algebras in $\mathcal{I}$ have stable rank one and $B \in \mathcal{I}$, there is an invertible $d_1' \in B$ such that

$$\|d_1 - d_1'\| < \varepsilon/8.$$

Let v be a partial isometry such that $v^*v = (1 - e_1 - p)$ and $vv^* \leqslant e_1$. Set $e_1' = vv^*$ and $d_2' = \varepsilon/8(e_1 - e_1') + (\varepsilon/8)v + (\varepsilon/8)v^* + d_2$. Note that $(\varepsilon/8)v + (\varepsilon/8)v^* + d_2$ has matrix decomposition

$$\begin{pmatrix} 0 & \varepsilon/8 \\ \varepsilon/8 & d_2 \end{pmatrix}.$$

Therefore d_2' is invertible in $(1 - p)A(1 - p)$. This implies that $d' = d_1' + d_2'$ is invertible in A. We also have

$$\|d_2' - d_2\| < \varepsilon/8,$$

whence

$$\|d - d'\| < \|d - (d_1 + d_2)\| + \|(d_1 + d_2) - (d_1' + d_2')\| < \varepsilon/16 + \varepsilon/8 + \varepsilon/8 < 3\varepsilon/8.$$

We have

$$\|b - u^*d'\| \leqslant \|u^*u(b - u^*d')\| = \|ub - d'\| < 3\varepsilon/8.$$

Finally,

$$\|a - u^*d'\| \leqslant \|a - b\| + \|b - u^*d'\| < \varepsilon/2 + 3\varepsilon/8 < \varepsilon.$$

Note that u^*d' is invertible. □

4.6. Corollary. *Every unital simple TAI C^*-algebra has the cancellation of projections, i.e., if $p \oplus e \sim q \oplus e$ then $p \sim q$.*

4.7. Theorem. *Every unital simple TAI C^*-algebra has the following Fundamental Comparability* [2]: *if $p, q \in A$ are two projections with $\tau(p) < \tau(q)$ for all tracial states τ on A, then $p \precsim q$.*

Proof. Denote by $T(A)$ the space of all normalized traces. It is compact. There is $d > 0$ such that $\tau(q-p) > d$ for all $\tau \in T(A)$. It follows from [31, 3.2] that there exists a nonzero projection $e \leqslant q$ such that $\tau(e) < d/2$ for all $\tau \in T(A)$. Set $q' = q - e$. Then $\tau(q'-p) > d/2$ for all $\tau \in T(A)$.

It follows from [8, 6.4] that there exists a nonzero $a \in A_+$ such that $q' - p - (d/4) = a + z$ and there is a sequence $\{u_n\}$ in A

$$z = \sum_n u_n^* u_n - \sum_n u_n u_n^*.$$

Choose an integer $N > 0$ such that

$$\left\| \sum_n u_n^* u_n - \sum_{n=1}^N u_n^* u_n \right\| < d/128 \quad \text{and} \quad \left\| \sum_n u_n u_n^* - \sum_{n=1}^N u_n u_n^* \right\| < d/128.$$

Let $\mathcal{F} = \{p, q, q', e, z, u_n, u_n^*, n = 1, 2, \ldots, N\}$ and let $0 < \varepsilon < 1$. Since A is TAI, there exists a projection $P \in A$ and a C^*-subalgebra $B \in \mathcal{I}$ with $1_B = P$ such that:

(1) $\|[x, P]\| < \varepsilon/2N$,
(2) $PxP \in_{\varepsilon/2N} B$ for all $x \in \mathcal{F}$ and
(3) $(1 - P) \precsim e$.

With sufficiently small ε, using a standard perturbation argument, we obtain projections $q'' = q_1 + q_2$, $p' = p_1 + p_2$, where q_1, q_2, p_1, p_2 are projections, $p_1, q_1 \in B$ and $q_2, p_2 \in (1-P)A(1-p)$ such that

$$\|q'' - q'\| < d/32 \quad \text{and} \quad \|p' - p\| < d/32.$$

Furthermore (with sufficiently small ε), we obtain $v_1, v_2, \ldots, v_N \in B$ such that

$$\left\| \big(q_1 - p_1 - (d/4)P\big) - \left(b + \sum_{n=1}^N v_n^* v_n - \sum_{n=1}^N v_n v_n^*\right) \right\| < d/16,$$

where $b \in B_+$ and $\|PaP - b\| < \varepsilon/2N$. Denote by $T(B)$ the space of all normalized traces on B. Then

$$\tau\big(q_1 - p_1 - (d/4)P - b\big) > -d/16$$

for all $\tau \in T(B)$. Therefore

$$\tau(q_1 - p_1) > d/4 - d/16 = 3d/16$$

for all $\tau \in T(B)$. This implies that $p_1 \precsim q_1$ in B, whence also in A. Since $p_2 \precsim (1 - P) \precsim e$, we conclude that

$$[p] = [p_1 + p_2] \leqslant [q_1] + [e] \leqslant [q]. \qquad \square$$

4.8. Theorem. *Let A be a unital simple TAI C^*-algebra. Then $K_0(A)$ is weakly unperforated and satisfies the Riesz interpolation property.*

Proof. First we note that, by 3.6, $M_n(A)$ is a unital simple TAI C^*-algebra. To show that $K_0(A)$ is weakly unperforated, it suffices to show that if $k[p] > k[q]$ for any projections in $M_n(A)$, then $[p] \geqslant [q]$, where $k > 0$ is an integer. But $k[p] > k[q]$ implies that $\tau(p) > \tau(q)$ for all traces. This implies that $[p] \geqslant [q]$ by 4.7. So $K_0(A)$ is weakly unperforated.

Since A has cancellation, to show that $K_0(A)$ has the Riesz interpolation property, it suffices to show the following. If $p \leqslant q$ are two projections in A and $q = q_1 + q_2$, where q_1 and q_2 are two mutually orthogonal projections, then $p = p_1 + p_2$ with $p_1 \leqslant q_1$ and $p_2 \leqslant q_2$. Without loss of generality, we may assume $q - p \neq 0$. Since $q_1 A q_1$ is a TAI C^*-algebra, there is a nonzero projection $q_1' \leqslant q_1$ such that $[p] \leqslant (q_1 - q_1') + q_2$ and $q_1 - q_1' \neq 0$. Let $q' = (q_1 - q_1') + q_2$. Then $p \precsim q'$. Without loss of generality, we may assume that $p \leqslant q'$.

Let $\mathcal{F}$ be a finite subset containing $p, q_1 - q_1', q_2$. For any $\varepsilon > 0$, there exists a unital C^*-subalgebra $B \in \mathcal{I}$ and a projection $P \in A$ with $1_B = P$ such that:

(1) $\|[x, P]\| < \varepsilon$,
(2) $PxP \in_\varepsilon B$ for all $x \in \mathcal{F}$ and
(3) $(1 - P) \precsim q_1'$.

With sufficiently small ε, without loss of generality, we may assume that $[p, P] = [q_1 - q_1', P] = [q_2, P] = 0$. Write $p'' = PpP$, $q_1'' = P(q_1 - q_1')P$ and $q_2'' = Pq_2P$. We have $p'' \leqslant q_1'' + q_2''$. Note that M_n and $M_n(C([0,1]))$ have the Riesz interpolation property. So B has the Riesz property. There are $p_1' \precsim q_1''$ and $p_2' \precsim q_2''$ such that $p_1' + p_2' = p''$. Since $p - p'' \leqslant (1 - P) \leqslant q_1'$, we let $p_1'' = p - p' + p_1'$. Then $p_1'' \precsim q_1' + q_1'' \leqslant q_1$. Now $p = p_1'' + p_2'$. □

4.9. Let A be a unital separable simple TAI C^*-algebra. We summarize some of its properties: (i) A has stable rank one; (ii) A has at least one tracial state; (iii) A has Fundamental Comparison property; (iv) A has weakly unperforated $K_0(A)$ and satisfies the Riesz interpolation property; (v) A has property (SP); (vi) A is MF; (vii) if A is nuclear, A is also quasidiagonal; (viii) $M_n(A)$ is TAI; (ix) Every quasitrace on A is a trace and $T(A)$ is a (metrizable) Choquet simplex; (x) $A \otimes F$ is TAI for all AF-algebras F; (xi) direct limits of TAI C^*-algebras are TAI and, in fact, locally TAI C^*-algebras are TAI.

We have not shown (ix). The only thing that one needs to note is that every quasitrace on C^*-algebras in $\mathcal{I}$ is in fact a trace. Then, from condition (3) of Definition 2.2, it is easy to see that every quasitrace is a trace. Note that it was proved in [3] that set of quasitraces on a unital C^*-algebra is a Choquet simplex.

We end this section with the following necessary and sufficient condition for a unital simple C^*-algebras to be TAI. For the simple case, one could use it as the definition.

4.10. Theorem. *Let A be a unital simple C^*-algebra. Then A is TAI if and only if the following hold. For any finite subset $\mathcal{F} \subset A$ containing a nonzero element b, $\varepsilon > 0$, integers $n > 0$ and $N > 0$, and any nonzero projection $e \in A$, there exist a nonzero projection $p \in A$ and a C^*-subalgebra $I = \bigoplus_{i=1}^k M_{n_i}(C([0,1]))$, with $1_I = p$ and $\min\{n_i\colon 1 \leqslant i \leqslant k\} \geqslant N$, such that:*

(1) $\|[x, p]\| < \varepsilon$ *for all* $x \in \mathcal{F}$,

(2) *$pxp \in_\varepsilon I$ for all $x \in \mathcal{F}$ and $\|pbp\| \geqslant \|b\| - \varepsilon$, and*
(3′) *$1 - p$ is unitarily equivalent to a projection in eAe.*

Proof. To show that the above is sufficient for A being TAI we note that A has property (SP) by 3.2. Then, by [31, 3.2], a result of Cuntz, there exists a projection $q \in eAe$ such that $(n+1)[q] \leqslant [e]$. Then it is clear that the above (3′) implies (3) in 2.2 (if we use the projection q instead of e).

To see it is also necessary, we use the fact that simple TAI C^*-algebras have stable rank one (so they have cancellation). It remains to show that we can make each summand of I have large rank. But this follows from (the proof of) 3.3. □

5. Tracial approximate divisibility and homomorphisms from C^*-algebras in $\mathcal{I}$

The main purpose of this section is to prove 5.8 and 5.9. Theorem 5.8 will be used to prove Theorem 8.6. Theorem 5.9 classifies monomorphisms from a C^*-algebra in $\mathcal{I}$ to a unital simple TAI C^*-algebra.

5.1. Sections 1–6 and 8 and most of 9 were written in a 1998 preprint titled "Classification of simple TAI C^*-algebras, part I" which was reported at the EU Operator Algebra Conference at Copenhagen in August 1998. The author later introduced the notation of tracial topological rank. When A is a unital simple C^*-algebra, A is a TAI C^*-algebra if and only if A has tracial topological rank no more than 1 (see [33, 7.1]).

The following is the definition of tracial topological rank no more than one for simple C^*-algebras.

5.2. Definition. Let A be a unital simple C^*-algebra. Then A has tracial topological rank no more than one, denote by $TR(A) \leqslant 1$, if the following holds. For any $\varepsilon > 0$, and any finite subset $\mathcal{F} \subset A$ containing a nonzero element $a \in A_+$, there is a C^*-subalgebra C in A with $C = \bigoplus_{i=1}^k M_{n_i}(C(X_i))$, where each X_i is a finite CW complex with dimension no more than one such that $1_C = p$ satisfying the following:

(i) $\|px - xp\| < \varepsilon$ for $x \in \mathcal{F}$,
(ii) $pxp \in_\varepsilon C$ for $x \in \mathcal{F}$ and
(iii) $1 - p$ is equivalent to a projection in $\overline{aAa}$.

In the above definition, if C can be chosen to be a finite-dimensional C^*-subalgebra then we write $TR(A) = 0$ (see [33]). If $TR(A) \leqslant 1$ but $TR(A) \neq 0$ (see [33]) then we will write $TR(A) = 1$.

In the light of [33, Theorem 7.1], *in what follows, we will replace unital simple TAI C^*-algebras by unital simple C^*-algebras with tracial topological rank no more than one and write $TR(A) \leqslant 1$.*

5.3. Definition. A unital simple C^*-algebra A is said to be *tracially approximately divisible* if for any $\varepsilon > 0$, any projection $e \in A$, any integer $N > 0$ and any finite subset $\mathcal{F} \subset A$, there exists a projection $q \in A$ and there exists a finite-dimensional C^*-subalgebra B with each simple summand having rank at least N such that:

(1) $\|qx - xq\| < \varepsilon$ for all $x \in \mathcal{F}$,

(2) $\|y(1-q)x(1-q)-(1-q)x(1-q)y]\| < \varepsilon$ for all $x \in \mathcal{F}$ and all $y \in B$ with $\|y\| \leqslant 1$, and
(3) q is unitarily equivalent to a projection of eAe.

Of course if A is approximately divisible, then A is tracially approximately divisible (see [7]).

5.4. Theorem. *Every nonelementary unital simple C^*-algebra with $TR(A) \leqslant 1$ is tracially approximately divisible.*

Proof. Let A be a unital simple C^*-algebra with $TR(A) \leqslant 1$. Fix $\varepsilon > 0$, $\sigma > 0$, $N > 0$ and a finite subset $\mathcal{F} \subset A$. Let $b \in A$ with $\|b\| = 1$ and assume that $b \in \mathcal{F}$. There exist a projection $p \in A$ and a C^*-subalgebra $C \in \mathcal{I}$ with $1_C = p$ such that:

(1) $\|px - xp\| < \varepsilon/4$ for all $x \in \mathcal{F}$,
(2) $pxp \in_{\varepsilon/4} C$ and $\|pbp\| \geqslant \|b\| - \varepsilon/2$, and
(3) $\tau(1-p) < \sigma/2$ for all traces τ on A.

Write $C = \bigoplus C_i$, where $C_i = M_{l(i)}(C[0,1])$, or $C_i = M_{l(i)}$. It will become clear that, without loss of generality, to simplify notation, we may assume that $C = C_i$ (i.e., there is only one summand). If $C = M_l$, let $\{e_{ij}\}$ be matrix units for M_l. Since A is not elementary, there is a positive element $a \in e_{11}Ae_{11}$ such that $\mathrm{sp}(a) = [0,1]$ (see [1, p. 6.1]). This implies that $C \subset M_l(C([0,1]))$. So, we may assume that $C = M_l(C([0,1]))$. Let $\mathcal{G}_1 \subset C$ be a finite subset such that

$$\mathrm{dist}(pxp, \mathcal{G}_1) < \varepsilon/4$$

for all $x \in \mathcal{F}$. Let $\mathcal{G}$ be a finite subset of C containing $\{e_{ij}\}$ and $e_{ij}ge_{ij}^*$ for all $g \in \mathcal{G}_1$.

Let $\eta > 0$. Denote by δ the positive number in Theorem 4.3 of [26] corresponding to η (instead of ε). Let $\{f_1, f_2, \ldots, f_m\} \subset C([0,1])$ be as in 3.1 with respect to δ $(= d)$. We identify $C([0,1])$ with $e_{11}Ce_{11}$. Since $e_{11}Ae_{11}$ is simple, there are $b_{ij} \in e_{11}Ae_{11}$ such that

$$\left\| \sum_{i=1} b_{ij} f_j b_{ij}^* - e_{11} \right\| < 1/16,$$

$j = 1, 2, \ldots, m$. Let $\mathcal{G}_2$ be a finite subset containing $\{f_j, b_{ij}, b_{ij}^*\} \bigcup \{a_{ij} \in e_{11}Ae_{11}\colon (a_{ij})_{l\times l} \in \mathcal{G}\}$.

By 3.4, $TR(e_{11}Ae_{11}) \leqslant 1$. So for any $0 < \sigma < \eta/2$ and any finite subset $\mathcal{G}_3 \supset \mathcal{G}_2$, there exist a projection $q \in e_{11}Ae_{11}$ and a C^*-subalgebra $C_1 \subset e_{11}Ae_{11}$ with $1_{C_1} = q$ and $C_1 \in \mathcal{I}$ satisfying the following:

(a) $\|qx - xq\| < \sigma$,
(b) $qxq \in_\sigma C_1$ for all $x \in \mathcal{G}_3$,
(c) $\tau(e_{11} - q) < \sigma/2l$ for all traces τ.

With sufficiently small σ and sufficiently large $\mathcal{G}_2$, we may assume that there exists a homomorphism $\phi : C([0,1]) \to C_1$ such that

(b′) $\|\phi(x) - qxq\| < \eta/2$ for all $x \in \mathcal{G}_2 \cap C([0,1])$.

Note that we also have $c_{ij} \subset C_1$ such that

$$\left\| \sum_{i=1} c_{ij}\phi(f_j)c_{ij}^* - q \right\| < 1/8, \quad j = 1, 2, \ldots, m.$$

We are now applying [26, Theorem 4.3]. It follows from 3.1 that $\mathrm{Sp}(\phi_t)$ is δ-dense in [0, 1]. By applying [26, 4.3], there is a homomorphism $\psi : C([0,1]) \to C_1$ and there is a finite-dimensional C^*-subalgebra $F = \bigoplus_i F_i$, where each F_i is simple and $\dim F_i \geqslant N$, with $1_F = q$ such that

$$\left\|\psi(f) - \phi(f)\right\| < \eta/2 \quad \text{for all } f \in \mathcal{G}_2 \quad \text{and}$$
$$\left\|\left[\psi(g), b\right]\right\| = 0$$

for all $g \in C([0,1])$ and $b \in F$. Set $F' = \mathrm{diag}(F, F, \ldots, F)$ in $F \otimes M_l$, $\psi' = \psi \otimes \mathrm{id}_{M_l}$, $\phi' = \phi \otimes \mathrm{id}_{M_l}$ and $P = \mathrm{diag}(q, q, \ldots, q) \in M_l(C_1)$. With sufficiently small η and large $\mathcal{G}_2$, we have

$$\left\|\psi'(g) - \phi'(g)\right\| < \varepsilon/2 \quad \text{for } g \in \mathcal{G}.$$

We also have

$$\left\|\left[\psi'(f), c\right]\right\| = 0 \quad \text{for } f \in C \text{ and } c \in F'.$$

These imply that

$$\left\|[PxP, c]\right\| < \varepsilon \quad \text{for all } x \in \mathcal{F} \text{ and } c \in F'.$$

Note that $1_{F'} = P$. We also have

$$\tau(1 - P) \leqslant \sigma/2 + l\sigma/2l = \sigma.$$

By 4.7, we conclude that A is tracially approximately divisible. When $C = \bigoplus C_i$, it is clear that we can do exactly the same as above for each summand. Let $d_i = 1_{C_i}$. If we find a matrix algebra $F_i \in d_i A d_i$ with rank greater than N which commutes with C_i, then $\bigoplus F_i$ commutes with C. □

5.5. Lemma. *Let A be a unital nuclear simple C^*-algebra with $TR(A) \leqslant 1$. Then for any $\varepsilon > 0$, any $\sigma > 0$, any integer $n > 0$, and any finite subset $\mathcal{F} \subset A$, there exist mutually orthogonal projections $q, p_1, p_2, \ldots, p_n$ with $q \precsim p_1$ and $[p_1] = [p_i]$ $(i = 1, 2, \ldots, n)$, a C^*-subalgebra $C \in \mathcal{I}$ with $1_C = p_1$ and completely positive linear contractions $L_1 : A \to qAq$ and $L_2 : A \to C$ such that*

$$\left\|x - \left(L_1(x) \oplus \mathrm{diag}\left(L_2(x), L_2(x), \ldots, L_2(x)\right)\right)\right\| < \varepsilon \quad \textit{and}$$
$$\left\|L_i(xy) - L_i(x)L_i(y)\right\| < \varepsilon,$$

where $L_2(x)$ is repeated n times, for all $x, y \in \mathcal{F}$ and $\tau(q) < \sigma$ for all $\tau \in T(A)$.

Proof. From the proof of 5.4, we have the following. For any $\eta > 0$, any integer $K > 0$, any integer $N > 4Kn^2$ and finite subset $\mathcal{G} \subset A$ (containing 1_A), there exists a projection $P \in A$ and a finite-dimensional C^*-subalgebra B with $1_B = P$ such that:

(i) $\|[P, x]\| < \eta$ for all $x \in \mathcal{G}$;
(ii) every simple summand of B has rank at least N;
(iii) there is a C^*-subalgebra $D \in \mathcal{I}$ with $1_D = P$ such that $[d, g] = 0$ for all $d \in D$, $g \in B$ and

$$\operatorname{dist}(x, D) < \eta \quad \text{for } x \in \mathcal{G}; \quad \text{and}$$

(iv) $5N[(1-P)] < [P]$.

Let $\mathcal{F}_1 \subset A$ be a finite subset (containing 1_A) and $\sigma > 0$. Since A is nuclear, with sufficiently large $\mathcal{G}$ and sufficiently small η, by [32, 3.2], there are unital completely positive linear contractions $L_1' : A \to (1-P)A(1-P)$ and $L_2' : A \to D$ such that $L_1'(a) = (1-P)a(1-P)$,

$$\|x - L_1'(x) \oplus L_2'(x)\| < \sigma \quad \text{and} \quad \|L_2'(x) - PxP\| < \eta + \sigma$$

for all $x \in \mathcal{F}_1$. It follows that, with sufficiently small σ and η,

$$\|L_i'(xy) - L_i'(x)L_i'(y)\| < \varepsilon$$

for all $x, y \in \mathcal{F}_1$. Write $B = \bigoplus_{i=1}^k B_i$, where $B_i \cong M_{l(i)}$ with $l(i) \geqslant N$, and denote by C the C^*-subalgebra generated by D and B. Note that $C \cong \bigoplus_{i=1}^k D_0 \otimes B_i$, where $D_0 \cong D$. Let $\pi_i : C \to D_0 \otimes B_i$ be the projection. Denote $\phi_i = \pi_i \circ L_2'$. By (iii), we see that we may write $\phi_i = \operatorname{diag}(\psi_i, \psi_i, \ldots, \psi_i)$, where $\psi_i : A \to e_i(D_0 \otimes M_{l(i)})e_i$ and e_i is a minimal rank-one projection of $M_{l(i)}$. Write $l(i) = k_i n + r_i$, where $k_i \geqslant n > r_i \geqslant 0$ are integers. We may rewrite

$$\phi_i = \operatorname{diag}(\Phi_i', \ldots, \Phi_i') \oplus \Psi_i',$$

where $\Phi_i' = \operatorname{diag}(\psi_i, \ldots, \psi_i) : A \to D_0 \otimes M_{k_i}$ is repeated n times and $\Psi_i' = \operatorname{diag}(\psi_i, \ldots, \psi_i) : A \to D_0 \otimes M_{r_i}$.

Define $L_2 = \bigoplus_{i=1}^k \Phi_i'$ and $L_1 = L_1' \bigoplus_{i=1}^k \Psi_i'$. We estimate that

$$\tau\left((1-P) + \bigoplus_{i=1}^k \Psi_i'(1_A)\right) < (1/5N)\tau(P) + (1/4nK)\tau(P) < (1/2n)\tau(P)$$
$$\leqslant \min\bigl(\sigma, \tau\bigl(\bigl[L_2(1_A)\bigr]\bigr)\bigr),$$

provided that $1/K < \sigma$. By 4.7, the lemma follows. □

The following corollary follows from Lemma 5.5 immediately.

5.6. Corollary. *Let A be a unital separable simple C^*-algebra $TR(A) \leqslant 1$. Then for any $\varepsilon > 0$, any $\sigma > 0$, any integer $n > 0$, and any finite subset $\mathcal{F} \subset A$, there exists a C^*-subalgebra $C \in \mathcal{I}$ such that*

$$\|x - (1-p)x(1-p) \oplus \operatorname{diag}(y, y, \ldots, y)\| < \varepsilon$$

where $y \in C$ and $\operatorname{diag}(y, y, \ldots, y) \in M_n(C)$ *and* $p = 1_{M_n(C)}$ *for all* $x, \in \mathcal{F}$ *and* $\tau((1-p)) < \sigma$ *for all* $\tau \in T(A)$. *Moreover, we may require that* $\|(1-p)x(1-p)\| \geqslant (1-\varepsilon)\|x\|$ *for all* $x \in \mathcal{F}$.

Proof. Perhaps the last part of the statement needs an explanation. In the proof of 5.5, we know that we may require that $\|y\| \geqslant (1-\varepsilon/2)\|x\|$ for all $x \in \mathcal{F}$. Thus we may replace $(1-p)x(1-p)$ by $(1-p)x(1-p) \oplus y$ and replace $(1-p)$ by $1 - p \oplus \operatorname{diag}(1_C, 0, \ldots, 0)$. □

5.7. Lemma. *Let* $B = \bigoplus_{i=1}^k B_i$ *be a unital* C^**-algebra in* $\mathcal{I}$ (*where* B_i *is a single summand*). *For any* $\varepsilon > 0$, *any finite subset* $\mathcal{F} \subset B$ *and any integer* $L > 0$, *there exist a finite subset* $\mathcal{G} \subset B$ *depending on* ε *and* $\mathcal{F}$ *but not on* L, *and* $\delta = 1/4L$ *such that the following holds. If* A *is a unital separable nuclear simple* C^**-algebra with* $TR(A) \leqslant 1$ *and* $\phi_i : B \to A$ *are two homomorphisms satisfying the following*:

(i) *there are* $a_{g,i}, b_{g,j} \in A$, $i, j \leqslant L$ *with*

$$\left\|\sum_i a_{g,i}^* \phi_1(g) a_{g,i} - 1_A\right\| < 1/16 \quad \text{and} \quad \left\|\sum_j b_{g,j}^* \phi_2(g) b_{g,j} - 1_A\right\| < 1/16$$

for all $g \in \mathcal{G}$;

(ii) $(\phi_1)_* = (\phi_2)_*$ *on* $K_0(B)$; *and,*

(iii) *if* $\|\tau \circ \phi_1(g) - \tau \circ \phi_2(g)\| < \delta$ *for all* $g \in \mathcal{G}$, *then there exists a unitary* $u \in A$ *such that*

$$\left\|\phi_1(f) - u^* \phi_2(f) u\right\| < \varepsilon \quad \text{for all } f \in \mathcal{F}.$$

Proof. It is clear that we can reduce the general case to the case in which B has only one summand. Since the case in which $B = M_{l(i)}$ is well known to hold, we may assume that $B = M_l(C([0,1]))$. Fix any $d_0 > 0$. Condition (i), with sufficiently large $\mathcal{G}$, implies that $\operatorname{Sp}(\phi_i)$ is d_0-dense in $[0,1]$ ($i = 1, 2$). By the proof of 3.7, therefore, for any $d_1 > d_0$, we may assume that

$$\phi_i(f) = \phi_i'(f) \oplus \sum_{j=1}^{N_i} f(t_{(i,j)}) q(i,j) \quad i = 1, 2,$$

where $\{t_{(i,1)}, t_{(i,2)}, \ldots, t_{(i,N_i)}\}$ ($i = 1, 2$) is d_1-dense in $[0,1]$ and $q(i,1), \ldots, q(i,N_i)$ are mutually (non-zero) orthogonal projections in A. It is clear that without loss of generality, we may assume that $t_j = t_{i,j}$ and $N_1 = N_2$. Since $TR(A) \leqslant 1$, we can find nonzero projections $q(i,j)' \leqslant q(i,j)$ such that $q(1,j)'$ and $q(2,j)$ are unitarily equivalent. By replacing ϕ_1 by $\operatorname{ad} z \circ \phi_1$ for some unitary z, we may assume that $q(1,j)' = q(2,j)'$. Then, by replacing ϕ_i' by $\phi_i' \oplus \sum_{j=1}^{N_i} f(t_j)(q(i,j) - q(i,j)')$, we may assume that, with $q_j = q(i,j)'$ and $N = N_1$,

$$\phi_i(f) = \phi_i'(f) \oplus \sum_{j=1}^{N} f(t_j) q_j \quad (i = 1, 2).$$

Now let $Q = 1 - \sum_{j=1}^N q_j$. We will apply [26, 5.14]. To do this, we let $r > 0$ be as in the statement of [26, 5.14] (but with respect to $\varepsilon/4$ and $\mathcal{L}_r \subset B$ (see also [26, 5.2])). Let $d = 1/r$ and $\delta = 1/4L$. Set $\mathcal{G} \subset \mathcal{L}_r$ such that the functions $f_1, f_2, \ldots, f_m$ required in 3.1 are all in $\mathcal{G}$.

Fix an integer $n > 1$. Let $e' \leqslant QAQ$ such that $n[e'] \leqslant [q_i]$, $i = 1, 2, \ldots, N$. This is possible since A is simple and has (SP). Let $\eta > 0$, $e \in QAQ$ be any nonzero projection in A with $\tau(e) < 1/2L$, $[e] \leqslant [e']$ and $K > 0$ be an integer. Since A is a unital simple C^*-algebra with $TR(A) \leqslant 1$, there exist a projection $P \in A$ and a unital C^*-subalgebra $C \in \mathcal{I}$ with $1_C = P$ such that:

(i) $\|[\phi_i'(g), P]\| < \eta$,
(ii) $P\phi_i(g)P \in_\eta C$ for all $g \in \mathcal{G}'$ and $i = 1, 2$, and
(iii) $K[Q - P] \leqslant [P]$ and $[Q - P] \leqslant [e]$,

where $\mathcal{G}' \supset \mathcal{G} \cup \{a_{g,i}, a_{g,i}^*, b_{g,j}, b_{g,j}^*, g \in \mathcal{G}$ and $i, j \leqslant L\}$. For any $\sigma > 0$, with sufficiently small η, there exists homomorphism $\psi_i : B \to C$ such that

$$\big\|\psi_i(g) - P\phi_i'(g)P\big\| < \sigma,$$

$$\left\|\sum_{i=1} c_{g,i}^* \psi_1(g) c_{g,i} - P\right\| < 1/8 \quad \text{and} \quad \left\|\sum_{j=1} d_{g,j}^* \psi_2(g) d_{g,j} - P\right\| < 1/8$$

for all $g \in \mathcal{G}$, where $c_{g,i}$, $d_{g,j} \in C$. It follows from 3.1 that

$$\big|\mathrm{sp}\big((\psi_i)_t\big) \cap T\big| \geqslant 1/L \big|\mathrm{Sp}\big((\psi_i)_t\big)\big|$$

for all $t \in [0, 1]$ (or, both ψ_1 and ψ_2 have the property $\mathrm{sdp}(r, 1/L)$ as in [26, 5.13]), where T has length at least $1/r$. We also have, if η and σ are sufficiently small,

$$\big\|\tau \circ \psi_1(g) - \tau \circ \psi_2(g)\big\| < 1/2L$$

for all $g \in \mathcal{G}$. It follows from 5.14 in [26] that there exists a unitary $v \in C$ such that

$$\big\|\psi_1(f) - v^* \psi_2(g) v\big\| < \varepsilon/4 \quad \text{for all } f \in \mathcal{F}.$$

We also have

$$\big\|\phi_i'(f) - \psi_i(f) - (Q - P)\phi_i'(f)(Q - P)\big\| < \varepsilon/4$$

for all $f \in \mathcal{F}$. Hence,

$$\left\|\phi_i(f) - \left(\sum_{j=1}^N f(t_j) q_j \oplus \psi_i'(f) \oplus (Q - P)\phi_i''(f)(Q - P)\right)\right\| < \varepsilon/2$$

for all $f \in \mathcal{F}$. Since

$$n[Q - P] \leqslant [q_i],$$

by, for example, [40, Lemma 8(i)] (this was known earlier), there exists (provided that n is sufficiently large and d_1 is sufficiently small, and these two numbers do not depend on ϕ_i or A) a unitary $w \in (1 - P)A(1 - P)$ such that

$$\left\|\sum_{j=1}^{N} f(t_j)q_j \oplus (Q-P)\phi_1''(f)(Q-P) - w^*\left(\sum_{j=1}^{N} f(t_j)q_j \oplus (Q-P)\phi_2''(f)\right)(Q-P)w\right\| < \varepsilon/2$$

for all $f \in \mathcal{F}$. Thus, we obtain a unitary $u \in A$ such that

$$\left\|\phi_1(f) - u^*\phi_2(f)u\right\| < \varepsilon \quad \text{for all } f \in \mathcal{F}. \qquad \square$$

5.8. Theorem. *Let A be a unital simple C^*-algebra with $TR(A) \leqslant 1$ and C be a C^*-subalgebra of A in $\mathcal{I}$. Then for any finite subset $\mathcal{F} \subset C$ and $\varepsilon > 0$, there exist $\delta > 0$, $\sigma > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: if $L_1, L_2 : A \to B$ are two unital $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps, where B is a unital simple C^*-algebra with $TR(B) \leqslant 1$, with $(L_1|_C)_* = (L_2|_C)_*$ on $K_0(C)$ and*

$$\left|\tau\big(L_1(g)\big) - \tau \circ L_2(g)\right| < \sigma$$

for all $g \in \mathcal{G}$ and for all $\tau \in T(B)$, then there is a unitary $u \in A$ such that

$$\left\|L_1(f) - u^*L_2(f)u\right\| < \varepsilon \quad \textit{for all } f \in \mathcal{F}.$$

Proof. Fix $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset A$. Let $\mathcal{G}_1 \subset C$ be the finite subset required by 5.7 (for a given $\varepsilon > 0$ and a given finite subset $\mathcal{F}$). Suppose that $a_{g,i} \in A$ such that

$$\left\|\sum_{i=1}^{n(g)} a_{g,i}^* g a_{g,i} - 1_A\right\| < 1/64$$

for all $g \in \mathcal{G}_1$. Set $L = \max\{n(g)\colon g \in \mathcal{G}\}$. Then, with sufficiently small $\delta > 0$ and large $\mathcal{G} \supset \mathcal{G}_1 \bigcup \{a_{g,i}\colon g, i\}$, we have $b_{g,i,j} \in A$ such that

$$\left\|\sum_{i=1}^{n(g)} b_{g,i,j}^* L_j(g) b_{g,i,j} - 1_B\right\| < 1/32$$

for all $g \in \mathcal{G}_1$ and $j = 1, 2$. Furthermore, for any $\eta > 0$, with sufficiently small δ, there is a homomorphism $\phi_j : C \to B$ ($j = 1, 2$) such that

$$\left\|\phi_j(g) - L_j(g)\right\| < \eta \quad \text{and} \quad \left\|\sum_{i=1}^{n(g)} b_{g,i,j}^* \phi_j(g) b_{g,i,j} - 1_B\right\| < 1/16$$

for $g \in \mathcal{G}_1$. We also require that $\sigma < 1/4L$. Then we see the conclusions of the theorem follow from 5.7 (and its proof) immediately. $\square$

5.9. Theorem. *Let A be a unital simple C^*-algebra with $TR(A) \leqslant 1$ and $B \in \mathcal{I}$. Let $\phi_i : B \to A$ be two monomorphisms such that*

$$(\phi_1)_* = (\phi_2)_* : K_0(B) \to K_0(A) \quad \textit{and} \quad \tau \circ \phi_1 = \tau \circ \phi_2$$

for all $\tau \in T(A)$. Then there is a sequence of unitaries $u_n \in A$ such that

$$\lim_{n\to\infty} u_n^* \phi_1(x) u_n = \phi_2(x) \quad \textit{for all } x \in B.$$

Proof. As before, we reduce the general case to the case in which $B = C([0, 1])$. Let $\varepsilon > 0$ and $\mathcal{F} \subset B$ be a finite subset. Let $\mathcal{G} \subset B$ be the finite subset in the statement of 5.7 (it does not depend on L). Since A is simple, there exists an integer $L > 0$ and $a_{i,g}, b_{i,g} \in A$, $i = 1, 2, \ldots, L$ (some of them could be zero) such that

$$\left\| \sum_i a_{i,g}^* \phi_1(g) a_{i,g} - 1 \right\| < 1/16 \quad \text{and} \quad \left\| \sum_i b_{i,g}^* \phi_2(g) b_{i,g} - 1 \right\| < 1/16$$

for all $g \in \mathcal{G}$. Therefore the theorem follows from 5.7. □

6. The unitary group of a simple C^*-algebra A with $TR(A) \leqslant 1$

We start with the following observations.

6.1. Let A be a unital C^*-algebra and $p, a \in A$. Suppose that p is a projection, $\|a\| \leqslant 1$ and

$$\|a^*a - p\| < 1/16 \quad \text{and} \quad \|aa^* - p\| < 1/16.$$

A standard computation shows that

$$\|pap - ap\| < 3/16 \quad \text{and} \quad \|pa - pap\| < 3/16.$$

Also $\|pa - a\| < 1/2$. Set $b = pap$. Then

$$\|b^*b - p\| \leqslant \|pa^*ap - pa^*a\| + \|pa^*a - p\| < 1/16 + 1/16 = 1/8.$$

So

$$\left\|(b^*b)^{-1} - p\right\| < \frac{1/8}{1 - 1/8} = 1/7 \quad \text{and} \quad \left\||b|^{-1} - p\right\| < 2/7,$$

where the inverse is taken in pAp. Set $v = b|b|^{-1}$. Then $v^*v = p = vv^*$ and

$$\|v - b\| < 2/7.$$

We denote v by $\tilde{a}$. Suppose that $L : A \to B$ is a $\mathcal{G}$-δ-multiplicative contractive completely positive linear map, u is a normal partial isometry and a projection $p \in B$ is given so that

$$\|L(u^*u)-p\| < 1/32.$$

Note if v' is another unitary in pAp with $\|v'-b\| < 1/3$, then $[v'] = [v]$ in $U(pAp)/U_0(pAp)$. We define $\tilde{L}$ as follows. Let $L(u) = a$. With small δ and large $\mathcal{G}$, we denote by $\tilde{L}(u)$ the normal partial isometry (unitary in a corner) v defined above. This notation will be used later. Note also, if $u \in U_0(A)$, then, with sufficiently large $\mathcal{G}$ and sufficiently small δ, we may assume that $\tilde{L}(u) \in U_0(B)$.

6.2. Definition. Let A be a unital C^*-algebra. Let $CU(A)$ be the *closure* of the commutator subgroup of $U(A)$. Clearly that the commutator subgroup forms a normal subgroup of $U(A)$. It follows that $CU(A)$ is a normal subgroup of A. It should be noted that $U(A)/CU(A)$ is commutative. It is an easy fact that if $A = M_r(C(X))$, where X is a finite CW complex of dimension 1, then $CU(A) \subset U_0(A)$. If $K_1(A) = U(A)/U_0(A)$, it is known and easy to verify that every commutator is in $U_0(A)$. Therefore $CU(A) \subset U_0(A)$. If $u \in U(A)$, we will use $\bar{u}$ for the image of u in $U(A)/CU(A)$, and if $F \subset U(A)$ is a subgroup of $U(A)$, then $\bar{F}$ is the image of F in $U(A)/CU(A)$.

If $\bar{u}, \bar{v} \in U(A)/CU(A)$ define

$$\operatorname{dist}(\bar{u}, \bar{v}) = \inf\bigl\{\|x-y\|\colon x, y \in U(A) \text{ such that } \bar{x} = \bar{u},\ \bar{y} = \bar{v}\bigr\}.$$

If $u, v \in U(A)$ then $\operatorname{dist}(\bar{u}, \bar{v}) = \inf\{\|uv^* - x\|\colon x \in CU(A)\}$. Let $g = \prod_{i=1}^n a_i b_i a_i^{-1} b_i^{-1}$, where $a_i, b_i \in U(A)$. Let $\mathcal{G}$ be a finite subset of A, $\delta > 0$ and $L : A \to B$ be a $\mathcal{G}$-δ-multiplicative contractive completely positive linear map, where B is a unital C^*-algebra. From 6.1, for $\varepsilon > 0$, if $\mathcal{G}$ is sufficiently large and δ is sufficiently small,

$$\left\| L(g) - \prod_{i=1}^n a_i' b_i' (a_i')^{-1} (b_i')^{-1} \right\| < \varepsilon/2,$$

where $a_i', b_i' \in U(B)$. Thus, for any $g \in CU(A)$, with sufficiently large $\mathcal{G}$ and sufficiently small δ,

$$\|L(g) - u\| < \varepsilon$$

for some $u \in CU(B)$. Moreover, for any finite subset $\mathcal{U} \subset U(B)$ and subgroup $F \subset U(B)$ generated by $\mathcal{U}$, and $\varepsilon > 0$, there exists a finite subset $\mathcal{G}$ and $\delta > 0$ such that, for any $\mathcal{G}$-δ-multiplicative contractive completely positive linear map $L : A \to B$, L induces a homomorphism $L^\ddagger : \bar{F} \to U(B)/CU(B)$ such that $\operatorname{dist}(\overline{\tilde{L}(u)}, L^\ddagger(\bar{u})) < \varepsilon$ for all $u \in \mathcal{U}$. Note we may also assume that $\bar{F} \cap U_0(A)/CU(A) \subset U_0(B)/CU(B)$.

If $\phi : A \to B$ is a homomorphism then $\phi^\ddagger : U(A)/CU(A) \to U(B)/CU(B)$ is the induced homomorphism. It is continuous.

Recall that, for a unitary $u \in U_0(A)$ in a unital C^*-algebra A, we write $\operatorname{cer}(u) \leqslant k$, if $u = \prod_{j=1}^k \exp(ih_j)$ for some self-adjoint elements $h_j \in A$. We write $\operatorname{cer}(u) \leqslant k + \varepsilon$ if u is a norm limit of unitaries u_n with $\operatorname{cer}(u_n) \leqslant k$.

Let $u \in U_0(A)$. Denote by $\operatorname{cel}(u)$ the infimum of the length of continuous paths of unitaries in A from u to 1_A.

6.3. Lemma *(N.C. Phillips). Let A be a unital C^*-algebra and $2 > d > 0$. Let $u_0, u_1, \dots, u_n$ be $n+1$ unitaries in A such that*

$$u_n = 1_A \quad \text{and} \quad \|u_i - u_{i+1}\| \leqslant d, \quad i = 0, 1, \dots, n-1.$$

Then there exists a unitary $v \in M_{2n+1}(A)$ with exponential length no more than 2π such that

$$\big\|\big(u_0 \oplus 1_{M_{2n}}(A)\big) - v\big\| \leqslant d.$$

Moreover, v can be chosen in $CU(M_{2n+1}(A))$.

The following is another version of the above lemma.

6.4. Lemma. *Let A be a unital C^*-algebra and $u \in U_0(A)$. Then for each $L > 0$, if $u = v \oplus (1-p)$ and $v \in U_0(pAp)$ with* $\mathrm{cel}(v) \leqslant L$ *in pAp and there are N $(> 2L)$ mutually orthogonal and mutually equivalent projections in $(1-p)A(1-p)$ each of which is equivalent to p, then* $\mathrm{cel}(u) \leqslant 2\pi + (L/n)\pi$. *Furthermore, there is a unitary $w \in CU(A)$ such that* $\mathrm{cel}(uw) < (L/n)\pi$.

(See the proof of [44, Theorem 3.8] and also [42, Corollary 5]. It should be noted that a unitary in $M_2(A)$ with the form $\mathrm{diag}(u, u^*)$ is in $CU(M_2(A))$.)

6.5. Theorem. *Let A be a unital simple C^*-algebra with $TR(A) \leqslant 1$. Let $u \in U_0(A)$. Then, for any $\varepsilon > 0$, there are unitaries $u_1, u_2 \in A$ such that u_1 has exponential length no more than 2π, u_2 is an exponential and*

$$\|u - u_1u_2\| < \varepsilon.$$

Moreover, $\mathrm{cer}(A) \leqslant 3 + \varepsilon$.

Proof. Let ε be a positive number. Let $v_0, v_1, \dots, v_n \in U_0(A)$ such that

$$v_0 = u, v_n = 1 \quad \text{and} \quad \|v_i - v_{i+1}\| < \varepsilon/16, \quad i = 0, 1, \dots, n-1.$$

Let $\delta > 0$. Since $TR(A) \leqslant 1$, there exists a projection $p \in A$ and a unital C^*-subalgebra $B \subset A$ with $B \in \mathcal{I}$ and with $1_B = p$ such that:

(1) $\|[v_i, p]\| < \delta$, $i = 0, 1, \dots, n$,
(2) $pv_ip \in_\delta B$, $0, 1, \dots, n$, and
(3) $2(n+1)[1-p] \leqslant p$.

There are unitaries $w_i \in (1-p)A(1-p)$ with $w_n = (1-p)$ such that

$$\big\|w_i - (1-p)v_i(1-p)\big\| < \varepsilon/16, \quad i = 0, 1, \dots, n$$

for any given $\varepsilon > 0$, provided δ is sufficiently small. Furthermore, there is a unitary $z \in B$ such that

$$\|z - pup\| < \varepsilon/16.$$

Therefore (with $\delta < \varepsilon/32$)

$$\|u - w_1 \oplus z\| < \varepsilon/8.$$

Write $z_1 = w_1 \oplus p$. Since $2(n+1)[1-p] \leqslant p$, by 6.3, there is a unitary u_1 with exponential length no more than 2π such that

$$\|z_1 - u_1\| < \varepsilon/4.$$

Now since $z \in B$ and it is well known that B has exponential rank $1 + \varepsilon$, there is an exponential $u_2 \in A$ such that

$$\left\|u_2 - (1-p) - z\right\| < \varepsilon/3.$$

Therefore

$$\|u - u_1 u_2\| < \varepsilon.$$

Since $\mathrm{cel}(u_1) \leqslant 2\pi$, it follows from [49] that $\mathrm{cer}(u_1) \leqslant 2 + \varepsilon$. Therefore $\mathrm{cer}(u) \leqslant 3 + \varepsilon$. So $\mathrm{cer}(A) \leqslant 3 + \varepsilon$. $\square$

6.6. Lemma. *Let A be a unital C^*-algebra.*

(1) *$U_0(A)/CU(A)$ is divisible.*
(2) *If $u \in U(A)$ such that $u^k \in U_0(A)$. Then there is $v \in U_0(A)$ such that $\bar{v}^k = \bar{u}^k$ in $U(A)/CU(A)$.*
(3) *Suppose that $K_1(A) = U(A)/U_0(A)$ and $G \subset U(A)/CU(A)$ is finitely generated subgroup. Then one has $G = G \cap (U_0(A)/CU(A)) \oplus \kappa(G)$, where $\kappa : U(A)/CU(A) \to U(A)/U_0(A)$ is the quotient map.*

Proof. Let $u \in U_0(A)$. Then there are $a_1, a_2, \ldots, a_n \in A_{\mathrm{sa}}$ such that $u = \prod_{j=1}^n \exp(ia_j)$. For any integer $k > 0$, let $v = \prod_{j=1}^n \exp(ia_j/k)$. Then $\bar{v}^k = \bar{u}$. This proves (1).

To see (2), put $u^k = \prod_{j=1}^n \exp(ia_j)$, where $a_j \in A_{\mathrm{sa}}$. Let $v = \prod_{j=1}^n \exp(ia_j/k)$. Thus $\overline{(uv^*)^k} = \bar{1}$. So $\bar{v}^k = \bar{u}^k$. To see (3), we note that (1) implies $0 \to U_0(A)/CU(A) \to G + U_0(A)/CU(A) \to \kappa(G) \to 0$ splits. $\square$

6.7. Theorem. *Let A be a unital simple C^*-algebra with $TR(A) \leqslant 1$ and $e \in A$ be a projection. Let $\gamma : U(eAe)/CU(eAe) \to U(A)/CU(A)$ be defined by $\gamma(\bar{u}) = \overline{u \oplus (1-e)}$. Then γ is a surjective (contractive) homomorphism.*

Proof. It is clear that γ is a homomorphism and is contractible. We will show that γ is also surjective. Fix $u \in U(A)$. Let $N > 0$ be an integer such that $N[e] \geqslant 1$ in A. Fix $1/2 > \varepsilon > 0$ and $0 < \eta < \varepsilon/8(N+1)$. It follows from 5.6 that there is a unitary $z_1 = s_0 \oplus s_1 \oplus s_1 \oplus \cdots \oplus s_1$, where

$s_0 \in U((e_0Ae_0))$ and $s_1 \in U(C)$ which repeats $n+1$ times ($n \geqslant 3$), where $C \in \mathcal{I}$ and $e_0 \oplus 1_C$ is equivalent to a subprojection of e such that

$$\|u - z_1\| < \eta/4.$$

Note that $M_{n+1}(C)$ is a C^*-subalgebra of A. By replacing s_0 by $s_0 \oplus s_1 \oplus \cdots \oplus s_1$, where s_1 repeats several times, we may assume that $3 \leqslant n \leqslant 4N+1$. Without loss of generality, we may also assume that $e_0 \oplus 1_C \leqslant e$. Let $w = e_0 \oplus s_1^n \oplus s_1^* \oplus s_1^* \oplus \cdots \oplus s_1^*$, where s_1^* repeats n times. Then $z_1 w = s_0 \oplus s_1^{n+1} \oplus 1_C \oplus 1_C \oplus \cdots \oplus 1_C$. Put $v_1 = s_0 \oplus s_1^{n+1} \oplus [e - (e_0 \oplus 1_C)]$ and put $y = s_1^n \oplus s_1^* \oplus s_1^* \oplus \cdots \oplus s_1^*$ (s_1^* repeats n times). Then $\det(y) = 1$ (in $M_{n+1}(C)$). Since $U_0(M_{n+1}(C)) = U(M_{n+1}(C))$, it follows from [54, 2.4] that $y \in CU(M_{n+1}(C))$. Hence $w \in CU(A)$. Therefore $\overline{v_1 \oplus (1-e)} = \bar{z}_1$.

Put $y_1 = z_1^* u$. Then $\|y_1 - 1_A\| < \eta/4$. We now repeat the same argument. We obtain $z_2 = s_0 \oplus s_1' \oplus \cdots \oplus s_1' \in U_0(A)$, where $s_0 \in U_0(e_0' A e_0')$ and where s_1' repeats $n+1$ times, $s_1' \in U_0(C_1)$, $C_1 \in \mathcal{I}$ and $e_0' \oplus 1_{C_1}$ is equivalent to a subprojection of e such that

$$\|z_2 - y_1\| < \eta/16.$$

Without loss of generality, we may further assume that $e_0' \oplus 1_{C_1} \leqslant e$. From the fact that $\|y_1 - 1_A\| < \eta/4$, we may assume that $\|s_0' - e_0'\| < \eta/2$ and $\|s_1' - 1_{C_1}\| < \eta/2$. Put $v_2 = s_0' \oplus (s_1')^{n+1} \oplus (e - (e_0' \oplus 1_{C_1}))$. Then (since $n < 4N+1$)

$$\|v_2 - e\| < \eta/2.$$

As we have shown, we have $\overline{v_2 \oplus (1-e)} = \bar{z}_2$. Note that $\overline{v_1 v_2 \oplus (1-e)} = \overline{z_1 z_2}$ and

$$\|z_1 z_2 - u\| \leqslant \|z_1 y_1 - u\| + \|z_1 y_1 - z_1 z_2\| = \|z_1 y_1 - z_1 z_2\| < \eta/16.$$

Also

$$\|v_1 v_2 - v_1\| < \eta/2.$$

Let $y_2 = (z_1 z_2)^* u$. Then $\|y_2 - 1_A\| < \eta/16$. We can continue the above argument. Consequently, we obtain a sequence of unitaries $z_n \in U(A)$ and a sequence of unitaries $v_n \in U(eAe)$ such that $\overline{v_1 v_2 \cdots v_n \oplus (1-e)} = \overline{z_1 z_2 \cdots z_n}$,

$$\|z_1 z_2 \cdots z_n - u\| \to 0 \quad \text{and} \quad \|v_1 v_2 \cdots v_n - v_1 v_2 \cdots v_m\| \to 0$$

as $n, m \to \infty$. Therefore we obtain a unitary $v \in U(eAe)$ such that

$$\overline{v \oplus (1-e)} = \bar{u}. \qquad \square$$

6.8. Lemma. *Let A be a unital C^*-algebra and $\mathcal{U} \subset U_0(A)$ be a finite subset. Then, for any $\varepsilon > 0$, there is a finite subset $\mathcal{G} \subset A$ and $\delta > 0$ satisfying the following: for any $\mathcal{G}$-δ-multiplicative contractive linear map $L : A \to B$ (for any unital C^*-algebra B), there are unitaries $v \in B$ such that*

$$\|L(u) - v\| < \varepsilon/2 \quad \textit{and} \quad \mathrm{cel}(v) < \mathrm{cel}(u) + \varepsilon/2$$

for all $u \in \mathcal{U}$.

Proof. Suppose that $z_0(u) = u, z_j(u) \in U_0(A),\ j = 1, 2, \dots, n(u)$ such that $\frac{\mathrm{cel}(u)}{n(u)} \leqslant 1/4$ and

$$\mathrm{cel}\big(z_j(u)\big(z_{j-1}(u)\big)^*\big) < \frac{\mathrm{cel}(u)}{n(u)}, \quad j = 1, 2, \dots, n(u),$$

for all $u \in \mathcal{U}$. Let $N = \max\{n(u)\colon u \in \mathcal{U}\}$. It follows that (for sufficiently large $\mathcal{G}$ and sufficiently small δ) there are unitaries $w_j(u) \in U(B)$ such that

$$\big\|L\big(z_j(u)\big) - w_j(u)\big\| < \varepsilon/8N\pi$$

for all j and $u \in \mathcal{U}$. Thus for all $u \in \mathcal{U}$,

$$\|L(u) - w_0(u)\| < \varepsilon/2\pi \quad \text{and} \quad \mathrm{cel}\big(w_0(u)\big) < n(u)\left[\frac{\mathrm{cel}(u)}{n(u)} + (\varepsilon/8N)2\pi\right] < \mathrm{cel}(u) + \varepsilon/2.$$

□

6.9. Lemma. *Let A be a unital simple C^*-algebra with $TR(A) \leqslant 1$ and let $u \in CU(A)$. Then $u \in U_0(A)$ and* $\mathrm{cel}(u) \leqslant 8\pi$.

Proof. We may assume that u is actually in the commutator group. Write $u = v_1 v_2 \cdots v_k$, where each v_i is a commutator. We write $v_i = a_i b_i a_i^* b_i^*$, where a_i and b_i are in $U(A)$. Fix integers $N > 0$ and $K > 0$. Since $TR(A) \leqslant 1$, by Corollary 3.3, there is a projection $p \in A$ and a C^*-subalgebra $B \in \mathcal{I}$ with $1_p = B$ and $B = \bigoplus_{i=1}^{l} M_{m_i}(C([0,1])) \bigoplus_{j=1}^{L} M_{n_i}$, where $m_i, n_i \geqslant K$ such that

$$\|a_i - (a_i' \oplus a_i')\| < \varepsilon/4k, \qquad \|b_i - (b_i' \oplus b_i'')\| < \varepsilon/4k, \quad i = 1, 2, \dots, k,$$

$$\left\| u - \left(\prod_{i=1}^{k} a_i' b_i' (a_i')^* (b_i')^* \oplus a_i'' b_i'' (a_i'')^* (b_i'')^* \right) \right\| < \varepsilon/8,$$

$a_i', b_i' \in U((1-p)A(1-p))$, $a_i'', b_i'' \in U_0(B)$ and $N[1-p] \leqslant [p]$. Put $w = \prod_{i=1}^{k} a_i' b_i' (a_i')^* (b_i')^*$ and $z = \prod_{i=1}^{k} a_i'' b_i'' (a_i'')^* (b_i'')^*$. Then $\mathrm{Det}(z) = 1$. It follows from [43, 3.4] (by choosing K large) we conclude that $\mathrm{cel}(z) \leqslant 6\pi$ in pAp. It is standard to show that $a_i' b_i' (a_i')^* (b_i')^* \oplus (1-p) \oplus (1-p)$ is in $U_0(M_4((1-p)A(1-p)))$ and it has exponential length no more than $4(2\pi) + \varepsilon/8k$. This implies that (in $U((1-p)A(1-p))$) $\mathrm{cel}(w \oplus (1-p)) \leqslant 8k\pi + \varepsilon/2$. Note the length only depends on k. We can then choose $N = N(8k\pi + \varepsilon)$ as in 6.4. In this way, $\mathrm{cel}(w \oplus p) \leqslant 2\pi + \varepsilon/2$. It follows that

$$\mathrm{cel}\big((w \oplus p)\big((1-p) \oplus z\big)\big) \leqslant 8\pi + \varepsilon/2.$$

The fact that $\|u - (w \oplus p)((1-p) \oplus z)\| < \varepsilon/8$ implies that $\mathrm{cel}(u) \leqslant 8\pi + \varepsilon$. □

6.10. Theorem. *Let A be a unital simple C^*-algebra with $TR(A) \leqslant 1$. Let $u, v \in U(A)$ such that $[u] = [v]$ in $K_1(A)$ and*

$$u^k, v^k \in U_0(A) \quad \text{and} \quad \mathrm{cel}\big(\big(u^k\big)^* v^k\big) < L.$$

Then

$$\mathrm{cel}\big(u^* v\big) \leqslant 8\pi + L/k.$$

Moreover, there is $y \in U_0(A)$ with $\mathrm{cel}(y) \leqslant L/k$ such that $\overline{u^* v} = \bar{y}$ in $U(A)/CU(A)$.

Proof. Suppose that

$$u^* v = \prod_j \exp(i a_j) \quad \text{and} \quad \big(u^k\big)^* v^k = \prod_m \exp(i b_m),$$

where $a_j, b_m \in A_{\mathrm{sa}}$. Since $\mathrm{cel}((u^k)^* v^k) < L$, we may assume that $\sum \|b_m\| < L$ (see [49]). Let $M = \sum_j \|a_j\|$. (So $\mathrm{cel}(u^* v) \leqslant M$.) Since $TR(A) \leqslant 1$, for any $\delta > 0$ with $\delta/(1-\delta) < \varepsilon/2(M + L + 1)$ and sufficiently small $\eta > 0$ and with a sufficiently large finite subset $\mathcal{G}$ (which contains a_j, b_m), there exist a projection $p \in A$ and a unital C^*-subalgebra $F \subset A$ with $1_F = p$ and $F \in \mathcal{I}$ such that:

(1) $pxp \in_\eta F$ for all $x \in \mathcal{G}$,
(2) $\|u - u_0 \oplus u_1\| < \eta$, $\|v - v_0 \oplus v_1\| < \eta$, and
(3) u_0 and v_0 are unitaries in $(1-p)A(1-p)$ and u_1 and v_1 are unitaries in pAp,
(4) $\mathrm{cel}(u_0^* v_0) \leqslant M + 1$ in $(1-p)A(1-p)$ and $\mathrm{cel}((u_1^k)^* v_1^k) < L$ in F,
(5) $\tau(1-p) < \delta$ for all $\tau \in T(A)$.

(Note (4) follows from 6.8).

Write $F = \bigoplus_s^N F_s$, where each $F_s = M_{n(s)}(C([0,1]))$ or $F_s = M_{n(s)}$. By Corollary 3.3, we may assume that each $n(s) > \max(2\pi^2/\varepsilon, K(1))$, where $K(1)$ is the number described in [43, Lemma 3.4] (with $d = 1$).

First consider the case in which $N = 1$ and $F = M_K(C([0,1]))$ (so $K > \max(2\pi^2/\varepsilon, K(1))$). Note that $\mathrm{cel}((u_1^*)^k v_1^k) < L$ in F. Therefore, by [43, Lemma 3.3(1)], there exists $a \in F_{\mathrm{sa}}$ with $\|a\| < L$ such that

$$\det\big(\exp(ia)\big(u_1^*\big)^k v_1^k\big) = 1 \quad \big(\text{for every } t \in [0,1]\big).$$

This implies that

$$\det\big(\big(\exp(ia/k) u_1^* v_1\big)^k\big) = 1 \quad \big(\text{for every } t \in [0,1]\big).$$

Therefore

$$\det\big(\exp(ia/k) u_1^* v_1\big) = \exp(i 2 l \pi/k) \quad \big(\text{for every } t \in [0,1]\big)$$

for some $l = 0, 1, \dots, k-1$. Note since determinant is a continuous function on $[0, 1]$, the above function has to be constant (only one value of l occurs). Set $f(t) = -2l\pi/k$. Then $f \in C([0, 1])_{sa}$ and $\|f\| \leqslant 2\pi$. Note that $\exp(if/K) \cdot 1_F$ commutes with $\exp(ia/k)$ and $(\exp(if/K) \cdot 1_F)\exp(ia/k) = \exp(i(f/K + a/k))$. We have

$$\det\bigl(\bigl(\exp(if/K) \cdot 1_F\bigr)\exp(ia/k)u_1^* v_1\bigr) = 1.$$

So, by [43, 3.4 (and 3.1)],

$$\operatorname{cel}\bigl(u_1^* v_1\bigr) \leqslant 2\pi/K + L/k + 6\pi \quad (\text{in } F).$$

Moreover, by [54, 2.4], $z_1 = (\exp(if/K) \cdot 1_F)\exp(ia/k)u_1^* v_1 \in CU(F)$. Note the above also holds when $F = M_K$. By considering each summand, the above also holds for the case in which $N > 1$.

Moreover, by Lemma 6.4, there is $y' \in CU(A)$ and $y'' \in U_0(A)$ such that $(u_0 \oplus p)^*(v_0 \oplus p) = y'y''$ and $\operatorname{cel}(y'') < \varepsilon/2$. Note that $((1-p) \oplus z_1)y' \in CU(A)$. Therefore

$$\overline{u^* v} = \overline{\exp(if/K) \cdot 1 \cdot \exp(ia/k)w}$$

for some $w \in U_0(A)$ with $\operatorname{cel}(w) < \varepsilon/2$ if η is sufficiently small. Therefore, by 6.9, with sufficiently small η,

$$\operatorname{cel}(u^* v) \leqslant 2\pi/K + L/k + 8\pi + \varepsilon/2 < 8\pi + L/k + \varepsilon. \qquad \square$$

6.11. Theorem. *Let A be a unital separable simple C^*-algebra with $TR(A) \leqslant 1$ and $u \in U_0(A)$. Suppose that $u^k \in CU(A)$ for some integer $k > 0$, then $u \in CU(A)$. In particular, $U_0(A)/CU(A)$ is torsion free.*

Proof. The proof is essentially the same as that of 6.10. Let $\varepsilon > 0$ and let

$$v = \prod_{j=1}^{r} a_i b_i \bigl(a_i'\bigr)^{-1}\bigl(b_i'\bigr)^{-1} \quad \text{be such that} \quad \bigl\|u^k - v\bigr\| < \varepsilon/64.$$

Put $l = \operatorname{cel}(u^k)$. Let $\delta > 0$ be such that $2(l + \varepsilon)\delta < \varepsilon/64\pi$. Fix a finite subset $\mathcal{G} \subset A$ which contains $u, u^k, v, a_i, b_i, a_i^{-1}, b_i^{-1}$ among other elements.

Since $TR(A) \leqslant 1$, there is a projection $p \in A$ and a unital C^*-subalgebra $F \in \mathcal{I}$ with $1_F = p$ such that:

(1) $pxp \in_{\varepsilon/64} F$ for all $x \in \mathcal{G}$,
(2) $\|v - v_0 \oplus v_1\| < \varepsilon/32$, $\|u - u_0 \oplus u_1\| < \varepsilon/32$, and $\|u^k - u_0^k \oplus u_1^k\| < \varepsilon/32$,
(3) $\operatorname{cel}(u_0^k) < l + \varepsilon/32$ in $U((1-p)A(1-p))$ and
(4) $\tau(1-p) < \delta$ for all $\tau \in T(A)$.

(Note (3) follows from 6.8 with large $\mathcal{G}$.)

Here $u_0, v_0 \in U((1-p)A(1-p))$ and $v_1, u_1 \in U(F)$. Moreover, we may assume that there are $a_i', b_i' \in U(F)$ such that

$$\left\| u_1^k - \prod_{j=1}^{r} a_i' b_i' \left(a_i'\right)^{-1} \left(b_i'\right)^{-1} \right\| < \varepsilon/32.$$

Put $w = \prod_{j=1}^{r} a_i' b_i' (a_i')^{-1} (b_i')^{-1}$. Since $U(F) = U_0(F)$, we may write $w = \prod_{m=1}^{L} \exp(i d_m)$ for some $d_m \in F_{\mathrm{sa}}$. Put $w_k = \prod_{m=1}^{L} \exp(i d_m/k)$. Then $w_k^k = w$. So

$$\operatorname{cel}\big((u_1)^k \big(w_k^*\big)^k\big) < \frac{\varepsilon\pi}{32}.$$

Write $F = \bigoplus_{s=1}^{N} F_s$, where each $F_s = M_{r(s)} C([0,1])$ or $F_s = M_{r(s)}$. By 3.3, we may assume that each $n(s) > \max(16\pi^2/\varepsilon, K(1))$, where $K(1)$ is the number described in [43, Lemma 3.4] (with $d = 1$).

As in the proof of 6.10,

$$\det\big(\exp(if/K)\exp(ia/k) u_1 w_k^*\big) = 1$$

for some $a, f \in F_{\mathrm{sa}}$ with $\|f\| \leqslant 2\pi$ and $\|a\| < \varepsilon\pi/32$ (with $K > \max(16\pi^2/\varepsilon, K(1))$). By [54, 2.4], $\exp(if/K)\exp(ia/k) u_1 w_k^* \in CU(F)$. We also have

$$\big\|\exp(if/K)\exp(ia/k) - 1_F\big\| < \varepsilon/8 + \varepsilon/32.$$

Thus

$$\operatorname{dist}\big(\overline{u_1 w_k^*}, \bar{1}\big) < \varepsilon/8 + \varepsilon/32.$$

Since $\det(w) = 1$, as in the proof of 6.10, we also have

$$\det\big(\exp(ig/K) w_k^*\big) = 1$$

for some $g \in F_{\mathrm{sa}}$ with $\|g\| \leqslant 2\pi$. Again, $\exp(ig/K) w_k^* \in CU(F)$. But

$$\big\|\big(\exp(ig/K) w_k\big) w_k^* - 1\big\| \leqslant \big\|\exp(ig/K) - 1\big\| < \varepsilon/4.$$

So

$$\operatorname{dist}(\overline{w_k}, \bar{1}) < \varepsilon/4.$$

Therefore

$$\operatorname{dist}(\bar{u}_1, \bar{1}) \leqslant \operatorname{dist}(\bar{u}, \overline{w_k}) + \operatorname{dist}(\overline{w_k}, \bar{1}) < \varepsilon/8 + \varepsilon/32 + \varepsilon/4 < \varepsilon/2$$

in $U(F)/CU(F)$. On the other hand, by 6.4 and the choice of δ,

$$\operatorname{cel}\big((u_0 \oplus p) z\big) < \varepsilon/(8\pi)$$

for some $z \in CU(A)$. Thus

$$\inf\bigl\{\|u_0 \oplus u_1 - x\|:\ x \in CU(A)\bigr\} < \varepsilon/8 + \varepsilon/8 + \varepsilon/32 + \varepsilon/4 < 3\varepsilon/4.$$

This implies that

$$\inf\bigl\{\|u - x\|:\ x \in CU(A)\bigr\} < \varepsilon.$$

Therefore $u \in CU(A)$. Consequently $U_0(A)/CU(A)$ is torsion free. □

6.12. Corollary. *Let B_n be a sequence of unital simple C^*-algebra with $TR(B_n) \leqslant 1$. Let $\prod_n^b K_1(B_n)$ be the set of sequences $z = \{z_n\}$, where $z_n \in K_1(B_n)$ and z_n can be represented by unitaries in $M_{K(z)}(B_n)$ for some integer $K(z) > 0$. Then the kernel of the map*

$$K_1\Bigl(\prod_n B_n\Bigr) \to \prod_n^b K_1(B_n) \to 0$$

is a divisible and torsion free subgroup of $K_1(\prod_n B_n)$.

Proof. By 6.5, the exponential rank of each B_n is bounded by 4. Therefore that the kernel is divisible follows from the fact that each B_n has stable rank one (and has exponential rank bounded by 4) (see [25]). Suppose that $\{u_n\} \in U(M_K(\prod_n B_n))$ such that $[\{u_n\}]$ is in the kernel and $k[\{u_n\}] = 0$. By changing notation (with different $\{u_n\}$ and larger K), we may assume that $\{u_n^k\} \in U_0(M_K(\prod_n B_n))$. Also each $u_n \in U_0(B_n)$. This implies that there is $L > 0$ such that $\mathrm{cel}(u_n^k) \leqslant L$ for all n. It follows from 6.10 that

$$\mathrm{cel}(u_n) \leqslant 8\pi + L + L/k + \pi/4 \quad \text{for all } n.$$

This implies (see for example [25]) $\{u_n\} \in U_0(M_L(\prod_n B_n))$. Therefore $[\{u_n\}] = 0$ in $K_1(\prod_n B_n)$. So the kernel is torsion free. □

6.13. Remark. When $\dim X \leqslant 1$, we believe that the conclusion of [43, 3.4] can be improved and $\mathrm{cel}(u)$ should be no more than $D(u) + 2\pi$. This could be achieved by a modification of Phillips' argument as we were informed by N.C. Phillips. Consequently, in 6.10, 8π could be replaced by 4π and in 6.9, 8π could be replaced by 4π. However, we will not use these better estimates.

7. Homomorphisms from $U(C)/CU(C)$ to $U(B)/CU(B)$

7.1. Definition. Let Y be a connected finite CW complex with dimension no more than three with torsion $K_1(C(Y))$ and set $C' = PM_r(C(X))P$, where $X = S^1 \vee S^1 \vee \cdots \vee S^1 \vee Y$ and $P \in M_r(C(X))$ is a projection and P has rank $R \geqslant 6$. We assume that S^1 is repeated s ($\geqslant 0$) times. Note that the above includes the case that $X = Y = [0, 1]$. Then $K_1(C') = tor(K_1(C')) \oplus G_1$, where G_1 is s copies of $\mathbf{Z}$. Denote by ξ the point in X where each S^1 and Y meet. Rename each S^1 by Ω_i, $i = 1, 2, \ldots, s$. Denote by z_i' the identity map on ith S^1 (Ω_i). Define $z_i''(\zeta) = \zeta$ on Ω_i (which is identified with S^1) and $z_i''(\zeta) = \xi$ for all $\zeta \notin \Omega_i$. There is an obvious homomorphism $\Pi : PM_r(C(X))P \to D' = \bigoplus_{i=1}^k E_i$, where $E_i \cong M_R(C(S^1))$. Note that if $s \geqslant 2$, then Π is not surjective. We have that $G_1 = K_1(D')$. We also use $\Pi_i : PM_r(C(X))P \to E_i$ which is the composition of Π with the projection from D' to E_i. Let z be the identity map on S^1.

We may write

$$P(t)=\begin{pmatrix} P_1 & 0 \\ 0 & I \end{pmatrix}$$

where P_1 is a projection with rank 3 and $I=\operatorname{diag}(1,1,\ldots,1)$ with 1 repeating $\operatorname{rank}(P)-3$ times.

Note that, since $\operatorname{rank}(P)\geqslant 6$, $\operatorname{tsr}(C(X))=2$ and $\operatorname{csr}(C(X))\leqslant 2+1$ (by 4.10 of [47]). It follows that $\operatorname{csr}M_3(C')\leqslant 2$ (by 4.7 in [48]). It then follows from 5.3 of [48] that $U(C')/U_0(C')=K_1(C')$. In particular, $CU(C')\subset U_0(C')$.

Denote by $u_i=\operatorname{diag}(z_i'',1,\ldots,1)$, where 1 is repeated $r-4$ times. If we write $z_i\in U(C)$, we mean the unitary

$$z_i(t)=\begin{pmatrix} P_1 & 0 \\ 0 & u_i \end{pmatrix}.$$

If we write $z_i\in E_i$, we mean $\Pi_i(z_i'')$. Note that in this case, z_i has the form $\operatorname{diag}(1,\ldots,1,z,1,\ldots,1)$, where z is in the 4th position and there are $R-1$ many 1's.

Now let $C=\bigoplus_{j=1}^{l+l_1}C^{(j)}$, where $C^{(j)}$ is either of the form $P_jM_{r(j)}(C(X_j))P_j$ for $j\leqslant l$, where X_j is of the form X described above, $C^{(j)}=M_{r(j)}$, or $C^{(j)}=P_jM_{r(j)}(C(Y_j))P_j$, where Y_j is a finite CW complex with dimension no more than 3, rank of P_j is $R(j)\geqslant 6$ and $K_1(Y_j)$ is finite for $l+1\leqslant j\leqslant l+l_1$. Let $D^{(j)}$ be as D' above for each $j\leqslant l$. Let $\Pi^{(j)}$ be as Π above for $C^{(j)}=P_jM_{r(j)}(C(X_j))P_j$. Put $D=\bigoplus_{j=1}^{l}D^{(j)}$ and $\Pi=\bigoplus_{j=1}^{l}\Pi^{(j)}$. Since $K_1(C)$ is finitely generated and $U_0(C)/CU(C)$ is divisible (see 6.6), we may write

$$U(C)/CU(C)=U_0(C)/CU(C)\oplus K_1(D)\oplus tor\bigl(K_1(C)\bigr).$$

Let $\pi_1:U(C)/CU(C)\to K_1(D)$, $\pi_0:U(C)/CU(C)\to U_0(C)/CU(C)$, $\pi_2:U(C)/CU(C)\to tor(K_1(C))$ be fixed projection maps associated with the above decomposition. To avoid possible confusion, by $\pi_i(U(C)/CU(C))$, we mean a subgroup of $U(C)/CU(C)$, $i=0,1,2$. We also assume that $\pi_1(\bar z_i)=\bar z_i$ (in $U(C)/CU(C)$).

It is worth pointing out that one could have $X=Y=[0,1]$.

The notation established above will be used in the rest of this section.

7.2. Lemma. *Let $C=\bigoplus_{i=1}^{l+l_1}C_i$ be as above and $\mathcal U\subset U(C)$ be a finite subset and F be the group generated by $\mathcal U$. Suppose that G is a subgroup of $U(C)/CU(C)$ which contains $\bar F$, $\pi_2(U(C)/CU(C)$ and $\pi_1(U(C)/CU(C))$. Suppose that the composition map $\gamma:\bar F\to U(D)/CU(D)\to U(D)/U_0(D)$ is injective and $\gamma(\bar F)$ is free. Let B be a unital C^*-algebra and $\Lambda:G\to U(B)/CU(B)$ be a homomorphism such that $\Lambda(G\cap U_0(C)/CU(C))\subset U_0(B)/CU(B)$. Then there are homomorphisms $\beta:U(D)/CU(D)\to U(B)/CU(B)$ with $\beta(U_0(D)/CU(D))\subset U_0(B)/CU(B)$, and $\theta:\pi_2(U(C)/CU(C))\to U(B)/CU(B)$ such that*

$$\beta\circ\Pi^{\ddagger}\circ\pi_1(\bar w)=\Lambda(\bar w)\bigl(\theta\circ\pi_2(\bar w)\bigr)$$

for all $w\in F$ and such that $\theta(g)=\Lambda|_{\pi_2(U(C)/CU(C))}(g^{-1})$ for $g\in\pi_2(U(C)/CU(C))$. Moreover, $\beta(U_0(D)/CU(D))\subset U_0(B)/CU(B)$.

If furthermore A is a simple C^*-algebra with $TR(B) \leqslant 1$ and $\Lambda(U(C)/CU(C)) \subset U_0(B)/CU(B)$, then $\beta \circ \Pi^\ddagger \circ (\pi_1)|_{\bar{F}} = \Lambda|_{\bar{F}}$.

Proof. Let $\kappa_1 : U(D)/CU(D) \to K_1(D)$ be the quotient map. Let $\eta : \pi_1(U(C)/CU(C)) \to K_1(D)$ be defined by $\eta = \kappa_1 \circ \Pi^\ddagger|_{\pi_1(U(C)/CU(C))}$. Note that η is an isomorphism. Since γ is injective and $\gamma(\overline{F})$ is free, we conclude that $\kappa_1 \circ \Pi^\ddagger \circ \pi_1$ is also injective on $\bar{F}$. From this fact and the fact that $U_0(C)/CU(C)$ is divisible (6.6), we obtain a homomorphism $\lambda : K_1(D) \to U_0(C)/CU(C)$ such that

$$\lambda|_{\kappa_1 \circ \Pi^\ddagger \pi_1(\bar{F})} = \pi_0 \circ \left(\left(\kappa_1 \circ \Pi^\ddagger \circ \pi_1\right)|_{\bar{F}}\right)^{-1}.$$

Now define $\beta = \Lambda((\eta^{-1} \circ \kappa_1) \oplus (\lambda \circ \kappa_1))$. Then for any $\bar{w} \in \bar{F}$,

$$\beta\left(\Pi^\ddagger \circ \pi_1(\bar{w})\right) = \Lambda\left[\eta^{-1}\left(\kappa_1 \circ \Pi^\ddagger\left(\pi_1(\bar{w})\right)\right) \oplus \lambda \circ \kappa_1\left(\Pi^\ddagger\left(\pi_1(\bar{w})\right)\right)\right] = \Lambda\left(\pi_1(\bar{w}) \oplus \pi_0(\bar{w})\right).$$

Now define $\theta : \pi_2(U(C)/CU(C)) \to U(B)/CU(B)$ by $\theta(x) = \Lambda(x^{-1})$ for $x \in \pi_2(U(C)/CU(C))$. Then

$$\beta\left(\Pi^\ddagger\left(\pi_1(\bar{w})\right)\right) = \Lambda(\bar{w})\theta\left(\pi_2(\bar{w})\right) \quad \text{for } w \in F.$$

To see the last statement, we assume $\Lambda(U(C)/CU(C)) \subset U_0(B)/CU(B)$. Then $\Lambda(\pi_2(U(C)/CU(C)))$ is a torsion subgroup of $U_0(B)/CU(B)$. By 6.11, $U_0(B)/CU(B)$ is torsion free. Therefore $\theta = 0$. □

7.3. Lemma. *Let A be a unital separable simple C^*-algebra with $TR(A) \leqslant 1$ and C be as described in* 7.1. *Let $\mathcal{U} \subset U(A)$ be a finite subset and F be the subgroup generated by $\mathcal{U}$ such that $(\kappa_1)|_{\bar{F}}$ is injective and $\kappa_1(\bar{F})$ is free, where $\kappa_1 : U(A)/CU(A) \to K_1(A)$ is the quotient map. Suppose that $\alpha : K_1(C) \to K_1(A)$ is an injective homomorphism and $L : \bar{F} \to U(C)/CU(C)$ is an injective homomorphism with $L(\bar{F} \cap U_0(A)/CU(A)) \subset U_0(C)/CU(C)$ such that $\pi_1 \circ L$ is injective* (*see* 7.1 *for* π_1) *and*

$$\alpha \circ \kappa_1' \circ L(g) = \kappa_1(g) \quad \textit{for all } g \in \bar{F},$$

where $\kappa_1' : U(C)/CU(C) \to K_1(C)$ is the quotient map. Then there exists a homomorphism $\beta : U(C)/CU(C) \to U(A)/CU(A)$ with $\beta(U_0(C)/CU(C)) \subset U_0(A)/CU(A)$ such that

$$\beta \circ L(\bar{w}) = \bar{w} \quad \textit{for } w \in F.$$

Proof. Let G be the preimage of $\alpha \circ \kappa_1'(U(C)/CU(C))$ under κ_1. So we have the following short exact sequence:

$$0 \to U_0(A)/CU(A) \to G \to \alpha \circ \kappa_1'\left(U(C)/CU(C)\right) \to 0.$$

Since $U_0(A)/CU(A)$ is divisible, there exists an injective homomorphism

$$\gamma : \alpha \circ \kappa_1'\left(U(C)/CU(C)\right) \to G$$

such that $\kappa_1 \circ \gamma(g) = g$ for $g \in \alpha \circ \kappa_1'(U(C)/CU(C))$. Since $\alpha \circ \kappa_1' \circ L(f) = \kappa_1(f)$ for all $f \in \bar{F}$, we have $\bar{F} \subset G$. Moreover, $(\gamma \circ \alpha \circ \kappa_1' \circ L(f))^{-1} f \in U_0(A)/CU(A)$ for all $f \in \bar{F}$. Define $\psi : L(\bar{F}) \to U_0(A)/CU(A)$ by

$$\psi(x) = \gamma \circ \alpha \circ \kappa_1' \circ L\big(\big[(L)^{-1}(x)\big]^{-1}\big) L^{-1}(x)$$

for $x \in L(\bar{F})$. Since $U_0(A)/CU(A)$ is divisible, there is homomorphism $\tilde{\psi} : U(C)/CU(C) \to U_0(A)/CU(A)$ such that $\tilde{\psi}|_{L(\bar{F})} = \psi$. Now define

$$\beta(x) = \gamma \circ \alpha \circ \kappa_1'(x)\tilde{\psi}(x).$$

Hence $\beta(L(f)) = f$ for $f \in \bar{F}$. $\square$

7.4. Lemma. *Let B be a unital separable simple C^*-algebra with $TR(B) \leqslant 1$ and C be as in 7.1. Let F be a group generated by a finite subset $\mathcal{U} \subset U(C)$ such that $(\pi_1)|_{\bar{F}}$ is injective. Let G be a subgroup containing $\bar{F}$, $\pi_0(\bar{F})$, $\pi_1(U(C)/CU(C))$ and $\pi_2(U(C)/CU(C))$. Suppose that $\alpha : U(C)/CU(C) \to U(B)/CU(B)$ is a homomorphism with $\alpha(U_0(C)/CU(C)) \subset U_0(B)/CU(B)$. Then for any $\varepsilon > 0$ there is $\delta > 0$ satisfying the following: if $\phi = \phi_0 \oplus \phi_1 : C \to B$ is a $\mathcal{G}$-η-multiplicative contractive completely positive linear map such that:*

(1) *both ϕ_0 and ϕ_1 are $\mathcal{G}$-η-multiplicative and ϕ_0 maps the identity of each summand of C into a projection,*
(2) *$\mathcal{G}$ is sufficiently large and η is sufficiently small which depend only on C and F (so that $\phi^{\ddagger}$ is well defined on a G),*
(3) *ϕ_0 is homotopically trivial (see (vi) in Section 1), $(\phi_0)_{*0}$ is a well-defined homomorphism and $[\phi]|_{\kappa_1(\bar{F})} = \alpha_*|_{\kappa_1(\bar{F})}$, where $\alpha_* : K_1(C) \to K_1(B)$ induced by α and $\kappa_1 : U(C)/CU(C) \to K_1(C)$ is the quotient map,*
(4) *$\tau(\phi_0(1_C)) < \delta$ for all $\tau \in T(B)$,*

then there is a homomorphism $\Phi : C \to e_0Be_0$ ($e_0 = \phi_0(1_C)$) such that:

(i) *Φ is homotopically trivial and $\Phi_{*0} = (\phi_0)_{*0}$ and*
(ii) $$\alpha(\bar{w})^{-1}(\Phi \oplus \phi_1)^{\ddagger}(\bar{w}) = \overline{g_w},$$
where $g_w \in U_0(B)$ and $\mathrm{cel}(g_w) < \varepsilon$ for all $w \in \mathcal{U}$.

Proof. By Lemma 7.2, there are homomorphisms $\beta_1, \beta_2 : U(D)/CU(D) \to U(B)/CU(B)$ with $\beta_i(U_0(D)/CU(D)) \subset U_0(B)/CU(B)$ ($i = 1, 2$) and homomorphisms

$$\theta_1, \theta_2 : \pi_2\big(U(C)/CU(C)\big) \to U(B)/CU(B) \quad \text{such that}$$
$$\beta_1 \circ \Pi^{\ddagger}\big(\pi_1(\bar{w})\big) = \alpha(\bar{w})\theta_1\big(\pi_2(\bar{w})\big) \quad \text{and} \quad \beta_2 \circ \Pi^{\ddagger}\big(\pi_1(\bar{w})\big) = \phi_1^{\ddagger}(\bar{w}^*)\theta_2\big(\pi_2(\bar{w})\big)$$

for all $\bar{w} \in \bar{F}$. Moreover $\theta_1(g) = \alpha(g^{-1})$ and $\theta_2(g) = \phi_1^{\ddagger}(g)$ if $g \in \pi_2(\bar{F})$. Since ϕ_0 is homotopically trivial,

$$\theta_1(g)\theta_2(g) \in U_0(B)/CU(B) \quad \text{for all } g \in \pi_2(\bar{F}).$$

Since $\pi_2(U(C)/CU(C)$ is torsion and $U_0(B)/CU(B)$ is torsion free, we conclude that

$$\theta_1(g)\theta_2(g) = \bar{1} \quad \text{for all } g \in \pi_2(\bar{F}).$$

To simplify notation, without loss of generality, we may assume that $C = \bigoplus_{j=1}^{1+l_1} C^{(j)}$ (with $l = 1$) such that $C^{(1)} = PM_r(C(X))P$ as described in 7.1 and $C^{(j)}$ is also as described in 7.1 for $2 = l + 1 \leqslant j \leqslant l_1 + 1$. Let D be as described in 7.1.

For each $w \in U(C)$, we may write $w = (w_1, w_2, \ldots, w_{1+l_1})$ according to the direct sum $C = \bigoplus_{j=1}^{1+l_1} C^{(j)}$. Note that $\pi_1(w) = \pi_1(w_1)$. Let $\pi_1(\bar{w}) = (\bar{z}_1^{k(1,w)}, \bar{z}_2^{k(2,w)}, \ldots, \bar{z}_s^{k(s,w)})$, where $k(i,w)$ is an integer (here z_i is described in 7.1). Then $\Pi_i^{\ddagger}(\pi_1(\bar{w}_1)) = \bar{z}_i^{k(i,w)}$. On the other hand, we may also write $\Pi_i^{\ddagger}(\bar{w}_1) = \overline{z_i^{k(i,w)} g_{i,w}}$ for some $g_{i,w} \in U_0(C(S^1, M_R))$.

Let $l = \max\{\mathrm{cel}(g_{i,w})\colon w \in \mathcal{U}, 1 \leqslant i \leqslant s\}$. Choose δ so that $(2+l)\delta < \varepsilon/4\pi$. Let $e_0 = \phi_0(1_C)$ and $e_1 = \phi_1(1_C)$. Write $e_0 = E_1 \oplus E_2 \oplus \cdots \oplus E_{1+l_1}$, where $E_j = \phi_0(1_{C^{(j)}})$, $j = 1, 2, \ldots, 1 + l_1$. Recall that P has rank R. Since ϕ_0 is homotopically trivial (see (vi) in Section 1), we may also write $E_1 = e_{01} \oplus \cdots \oplus e_{0R}$, where $\{e_{0i} : 1 \leqslant i \leqslant R\}$ is a set of mutually orthogonal and mutually equivalent projections. Since e_0Be_0 is simple and has the property (SP), e_{01} can be written as a sum of s mutually orthogonal projections. Thus $E_1 = p_1 \oplus p_2 \oplus \cdots \oplus p_s$, where each p_i can be written as a direct sum of R mutually orthogonal and mutually equivalent projections $\{q_{i,1}, \ldots, q_{i,R}\}$. For each q_{i1}, we write $q_{i1} = q_{i1,1} \oplus q_{i1,2}$, where both $q_{i1,1}$ and $q_{i1,2}$ are not zero. Let $q = \sum_{i=1}^{s} q_{i1}$. We may view $E_1BE_1 = \bigoplus_{i=1}^{s} M_R(q_{i1}Bq_{i1}) = M_R(qBq)$.

Let z_i be as in 7.1. Put $x_i' \in U(q_{i1,1}Bq_{i1,1})$ such that $\overline{x_i'} = \beta_1(\bar{z}_i)$, and $y_i' \in U(q_{i1,2}Bq_{i1,2})$ such that $\overline{y_i'} = \beta_2(\overline{z_i})$, $i = 1, 2, \ldots, s$. This is possible because of 6.7. Put $x_i = x_i' \oplus q_{i1,2}$, $y_i = y_i' \oplus q_{i1,1}$, $i = 1, 2, \ldots, s$. Note that $x_iy_i = y_ix_i$. Define $\Phi_1 : D \to M_R(qBq) = \bigoplus_{i=1}^{s} M_R(q_{i1}Bq_{i1})$ by $\Phi_1(f) = \sum_{i=1}^{s} f_i(x_iy_i)$, where $f = (f_1, f_2, \ldots, f_s)$, $f_i \in C(S^1, M_R)$. Define $h(g) = \Phi_1(\Pi(g)) \oplus \phi_1(g)$ for $g \in C$. We compute that

$$\begin{aligned}
\overline{h(w)} &= \prod_{i=1}^{s} \overline{x_i^{k(i,w)} g_{i,w}(x_iy_i) y_i^{k(i,w)}} \phi_1^{\ddagger}(\bar{w}) \\
&= \beta_1\bigl(\Pi^{\ddagger}\bigl(\pi_1(\bar{w})\bigr)\bigr)\Phi_1^{\ddagger}\left(\overline{\bigoplus_{i=1}^{s} g_{i,w}}\right)\beta_2\bigl(\Pi^{\ddagger}\bigl(\pi_1(\bar{w})\bigr)\bigr)\phi_1^{\ddagger}(\overline{w}) \\
&= \alpha(\bar{w})\theta_1\bigl(\pi_2(\bar{w})\bigr)\theta_2\bigl(\pi_2(\bar{w})\bigr)\Phi_1^{\ddagger}\left(\overline{\bigoplus_{i=1}^{s} g_{i,w}}\right) = \alpha(\bar{w})\Phi_1^{\ddagger}\left(\overline{\bigoplus_{i=1}^{s} g_{i,w}}\right)
\end{aligned}$$

for all $w \in \mathcal{U}$. Put $g_w' = \Phi_1(\bigoplus_{i=1}^{s} g_{i,w}) \oplus (1 - \phi_0(1_C))$. Since $\tau(\phi_0(1_C)) < \delta$, by the choice of δ, we conclude from Lemma 6.4 that there exists $w' \in CU(B)$ such that

$$\mathrm{cel}\bigl(w'g_w'\bigr) < \varepsilon/2 \quad \text{for all } w \in \mathcal{U}.$$

Note that $\Phi_1 \circ \Pi$ factors through D and $(\Phi_1)_{*1} = 0$. In particular, $\Phi_1 \circ \Pi$ is homotopically trivial. Since ϕ_0 is homotopically trivial, it is easy to see that there is a point-evaluation map $\Phi_2 : \bigoplus_{j=2}^{1+l_1} C^{(j)} \to (\bigoplus_{j=2}^{1+l_1} E_j)B(\bigoplus_{j=2}^{1+l_1} E_j)$. Now define $\Phi = \Phi_1 \circ \Pi \oplus \Phi_2$. We see that we can make (by a right choice of Φ_2) $\Phi|_{*0} = (\phi_0)_{*0}$. It is clear that Φ is homotopically trivial. Let

$g''_w = \Phi_2(w) \oplus (1 - (e_0 - E_1))$. Since $\Phi_2(\sum_{j=2}^{1+l_1} C^{(j)})$ is finite-dimensional, $\mathrm{cel}(\Phi_2(w)) \leqslant 2\pi$ (in $U_0((e_0 - E_1)B(e_0 - E_1))$ for all $w \in \mathcal{U}$). By the choice of δ, we conclude that there is $w'' \in CU(B)$ such that $\mathrm{cel}(w'' g''_w) < \varepsilon/2$ (see 6.4). Put $g_w = w' g'_w w'' g''_w$. We have, for all $w \in \mathcal{U}$,

$$\alpha(\bar{w})^{-1}(\Phi \oplus \phi_1)^{\ddagger}(\bar{w}) = \overline{g_w} \quad \text{with } g_w \in U_0(B) \text{ and } \mathrm{cel}(g_w) < \varepsilon. \qquad \Box$$

7.5. Lemma. *Let B be a unital separable simple C^*-algebra with $TR(B) \leqslant 1$ and C be as described in* 7.1. *Let $\mathcal{U} \subset U(B)$ be a finite subset and F be the subgroup generated by $\mathcal{U}$ such that $\kappa_1(\bar{F})$ is free, where $\kappa_1 : U(B)/CU(B) \to K_1(B)$ is the quotient map. Let $\phi : C \to B$ be a homomorphism such that ϕ_{*1} is injective. Suppose that $j, L : \bar{F} \to U(C)/CU(C)$ are two injective homomorphisms with $j(\bar{F} \cap U_0(B)/CU(B)), L(\bar{F} \cap U_0(B)/CU(B)) \subset U_0(C)/CU(C)$ such that $\kappa_1 \circ \phi^{\ddagger} \circ L = \kappa_1 \circ \phi^{\ddagger} \circ j = \kappa_1|_{\bar{F}}$ and all three are injective.*

Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\phi = \phi_0 \oplus \phi_1 : C \to B$, where ϕ_0 and ϕ_1 are homomorphisms satisfying the following:

(1) $\tau(\phi_0(1_C)) < \delta$ *for all $\tau \in T(B)$ and*
(2) ϕ_0 *is homotopically trivial,*

then there is a homomorphism $\psi : C \to e_0 B e_0$ ($e_0 = \phi_0(1_C)$) such that:

(i) $[\psi] = [\phi_0]$ *in $KL(C, B)$ and*
(ii) $(\phi^{\ddagger} \circ j(\bar{w}))^{-1}(\psi \oplus \phi_1)^{\ddagger}(L(\bar{w})) = g_w$, *where $g_w \in U_0(B)$ and* $\mathrm{cel}(g_w) < \varepsilon$ *for all $w \in \mathcal{U}$.*

Proof. The first part of the proof is essentially the same as that of 7.3. Let $\kappa'_1 : U(C)/CU(C) \to K_1(C)$ be the quotient map and let G be the preimage of $\phi_{*1} \circ \kappa'_1(U(C)/CU(C))$ under κ_1. Since $U_0(B)/CU(B)$ is divisible, there exists an injective homomorphism $\gamma : \phi_{*1} \circ \kappa'_1(U(C)/CU(C)) \to G$ such that $\kappa_1 \circ \gamma(g) = g$ for $g \in \phi_{*1} \circ \kappa'_1(U(C)/CU(C))$. Since $\phi_{*1} \circ \kappa'_1 \circ L(f) = \kappa_1(f) = \kappa_1(\phi^{\ddagger} \circ j(f))$ for all $f \in \bar{F}$, we have $\bar{F} \subset G$. Moreover,

$$\left[\gamma \circ \phi_{*1} \circ \kappa'_1 \circ L(f)\right]^{-1} \phi^{\ddagger} \circ j(f) \in U_0(B)/CU(B)$$

for all $f \in \mathcal{F}$. Define $\psi : L(\bar{F}) \to U_0(B)/CU(B)$ by

$$\psi(x) = \left[\gamma \circ \phi_{*1} \circ \kappa'_1(x)\right]^{-1}\left[\phi^{\ddagger} \circ j \circ L^{-1}(x)\right]$$

for all $x \in L(\bar{F})$. Since $U_0(B)/CU(B)$ is divisible, there is a homomorphism

$$\tilde{\psi} : U(C)/CU(C) \to U_0(B)/CU(B) \quad \text{such that} \quad \tilde{\psi}|_{L(\bar{F})} = \psi.$$

Define $\alpha : U(C)/CU(C) \to U(B)/CU(B)$ by $\alpha(x) = \gamma \circ \phi_{*1} \circ \kappa'_1(x)\tilde{\psi}(x)$ for all $x \in U(C)/CU(B)$. Note that

$$\alpha\big(L(f)\big) = \phi^{\ddagger} \circ j(f) \quad \text{for all } f \in \bar{F}.$$

Now the lemma follows from 7.4. $\Box$

8. A uniqueness theorem and automorphisms on simple C^*-algebras with $TR(A) \leqslant 1$

8.1. Definition. Let A and B be C^*-algebras. Two homomorphisms $\phi, \psi : A \to B$ are said to be *stably unitarily equivalent* if for any monomorphism $h : A \to B$, $\varepsilon > 0$ and finite subset $\mathcal{F} \subset A$, there exists an integer $n > 0$ and a unitary $U \in M_{n+1}(\tilde{B})$ (or in $M_{n+1}(B)$, if B is unital) such that

$$\left\| U^* \operatorname{diag}\big(\phi(a), h(a), h(a), \ldots, h(a)\big)U - \operatorname{diag}\big(\psi(a), h(a), h(a), \ldots, h(a)\big)\right\| < \varepsilon$$

for all $a \in \mathcal{F}$, where $h(a)$ is repeated n times on both diagonals.

Let A and B be C^*-algebras and $\phi, \psi : A \to B$ be (linear) maps. Let $\mathcal{F} \subset A$ and $\varepsilon > 0$. We write

$$\phi \sim_\varepsilon \psi \quad \text{on } \mathcal{F},$$

if there exists a unitary $u \in B$ such that

$$\left\| \operatorname{ad}(u) \circ \phi(a) - \psi(a) \right\| < \varepsilon \quad \text{for all } a \in \mathcal{F}.$$

We write

$$\phi \approx_\varepsilon \psi \quad \text{on } \mathcal{F}, \quad \text{if } \left\|\phi(a) - \psi(a)\right\| < \varepsilon \text{ for all } a \in \mathcal{F}.$$

8.2. Definition. Let A be a C^*-algebra.

(i) Denote by $\mathbf{P}(A)$ the set of all projections and unitaries in $M_\infty(\widetilde{A \otimes C_n})$, $n = 1, 2, \ldots,$ where C_n is an abelian C^*-algebra so that

$$K_i(A \otimes C_n) = K_*(A; \mathbf{Z}/n\mathbf{Z}).$$

One also has the following exact sequence:

$$\begin{array}{ccccc}
K_0(A) & \longrightarrow & K_0(A, \mathbf{Z}/k\mathbf{Z}) & \longrightarrow & K_1(A) \\
\uparrow \mathbf{k} & & & & \downarrow \mathbf{k} \\
K_0(A) & \longleftarrow & K_1(A, \mathbf{Z}/k\mathbf{Z}) & \longleftarrow & K_1(A)
\end{array}$$

(see [53]). As in [12], we use the notation

$$\underline{K}(A) = \bigoplus_{i=0,1,n\in\mathbf{Z}_+} K_i(A; \mathbf{Z}/n\mathbf{Z}).$$

By $\operatorname{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ we mean all homomorphisms from $\underline{K}(A)$ to $\underline{K}(B)$ which respect to the direct sum decomposition and the so-called Bockstein operations (see [12]). Denote by $\operatorname{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))^{++}$ those $\alpha \in \operatorname{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ with the property that $\alpha(K_0(A)_+ \setminus \{0\}) \subset K_0(B)_+ \setminus \{0\}$. It follows from [12] that if A satisfies the Universal Coefficient Theorem, then $\operatorname{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)) \cong KL(A, B)$. Moreover, one has the following short exact sequence,

$$0 \to \mathrm{Pext}\big(K_*(A), K_*(B)\big) \to KK(A,B) \to KL(A,B) \to 0.$$

A separable C^*-algebra A is said to satisfy Approximate Universal Coefficient Theorem (AUCT) if

$$KL(A,B) \cong \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B)\big)$$

for any σ-unital C^*-algebra B (see [39]). A separable C^*-algebra A which satisfies the UCT must satisfy the AUCT. If A satisfies the AUCT, for convenience, we will use $KL(A,B)^{++}$ for $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))^{++}$.

(ii) Let $L : A \to B$, be a contractive completely positive linear map. We also use L for the extension from $A \otimes \mathcal{K} \to B \otimes \mathcal{K}$ as well as maps from $\widetilde{A \otimes C_n} \to \widetilde{B \otimes C_n}$ for all n. Given a projection $p \in \mathbf{P}(A)$, if $L : A \to B$ is an $\mathcal{F}$-δ-multiplicative contractive completely positive linear map with sufficiently large $\mathcal{F}$ and sufficiently small δ, $\|L(p) - q'\| < 1/4$ for some projection q'. Define $[L](p) = [q']$ in $\underline{K}(B)$. It is easy to see this is well defined (see [30]). Suppose that q is also in $\mathbf{P}(A)$ with $[q] = k[p]$ for some integer k. By adding sufficiently many elements (partial isometries) in $\mathcal{F}$, we can assume that $[L](q) = k[L](p)$. Similarly as in 6.1, one can do the same for unitaries. Let $\mathcal{P} \subset \mathbf{P}(A)$ be a finite subset. We say $[L]|_{\mathcal{P}}$ is well defined if $[L](p)$ is well defined for every $p \in \mathcal{P}$ and if $[p'] = [p]$ and $p' \in \mathcal{P}$, then $[L](p') = [L](p)$. This always occurs if $\mathcal{F}$ is sufficiently large and δ is sufficiently small. In what follows we write $[L]|_{\mathcal{P}}$ when $[L]$ is well defined on $\mathcal{P}$.

(iii) Let $A = \bigoplus_{i=1}^n A_i$, where each A_i is a unital C^*-algebra. Suppose that $L : A \to B$ is a $\mathcal{G}$-ε-multiplicative contractive completely positive linear map. For any $\eta > 0$, if $\mathcal{G}$ is large enough and ε is small enough, we may assume that

$$\big\|L(1_{A_i}) - p_i\big\| < \eta, \qquad \big\|p_j L(1_{A_i})\big\| < \eta \quad \text{and} \quad \big\|L(1_{A_i}) p_j\big\| < \eta$$

for some projection $p_i \in B$ and $i \neq j$. Let $b = p_1 L(1_{A_1}) p_1$. Then, with sufficiently small η, we may assume that b is invertible in $p_1 B p_1$. Define $L_1(a) = b^{-1/2} p_1 L(a) p_1 b^{-1/2}$. Then $L_1(1_{A_1}) = p_1$. Consider $(1 - p_1) L (1 - p_1)$. It is clear that, for any $\delta > 0$, by induction and choosing a sufficiently large $\mathcal{G}$ and sufficiently small η and ε,

$$\|L - \Psi\| < \delta,$$

where $\Psi(a) = \bigoplus_{i=1}^n L_i(1_{A_i} a 1_{A_i})$ for $a \in A$. So, to save notation in what follows, we may assume that $L = \bigoplus_{i=1}^n L_i$, where each $L_i : A_i \to B$ is a completely positive contraction which maps 1_{A_i} to a projection in B and $\{L_1(1_{A_1}), L_2(1_{A_2}), \ldots, L_n(1_{A_n})\}$ are mutually orthogonal.

Throughout the rest of this section, $\mathbf{A}$ *denotes the class of separable nuclear C^*-algebras satisfying the Approximate Universal Coefficient Theorem.*

8.3. Lemma. *(See [35, Theorem 4.4].) Let B be a unital C^*-algebra and let A be a unital C^*-algebra in* $\mathbf{A}$ *which is a unital C^*-subalgebra of B. Let $\alpha : A \to B$ and $\beta : A \to B$ be two homomorphisms. Then α and β are stably approximately unitarily equivalent if $[\alpha] = [\beta]$ in $KK(A,B)$ and if A is simple or B is simple.*

The following is a modification of [35, Theorem 4.8]. A proof was given in the earlier version of this paper. Since then a more general version of the following appeared in [39, Theorem 3.9]. We will omit the original proof and view the following as a special case.

8.4. Theorem. *(Cf. [35, Theorem 4.8].) Let B be a C^*-algebra with stable rank one and $\mathrm{cel}(M_m(B)) \leqslant k$ for some $k \geqslant \pi$ and for all m, and let A be a unital simple C^*-algebra in $\mathbf{A}$ which is a C^*-subalgebra of B. Let $\alpha : A \to B$ and $\beta : A \to B$ be two homomorphisms. Then α and β are stably approximately unitarily equivalent if $[\alpha] = [\beta]$ in $KL(A, B)$.*

The following uniqueness theorem is a modification of [35, Theorem 5.3].

8.5. Theorem. *(See [35, 5.3].) Let A be a unital simple C^*-algebra in $\mathbf{A}$ and $\mathbf{L} : U(M_\infty(A)) \to \mathbf{R}_+$ be a map. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$ there exist a positive number $\delta > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{P} \subset \mathbf{P}(A)$ and an integer $n > 0$ satisfying the following: for any unital simple C^*-algebra B with $TR(B) \leqslant 1$, if $\phi, \psi, \sigma : A \to C$ are three $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps with*

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}},$$
$$\mathrm{cel}\big(\tilde{\phi}(u)^* \tilde{\psi}(u)\big) \leqslant \mathbf{L}(u)$$

for all $u \in U(A) \cap \mathcal{P}$ and σ is unital, then there is a unitary $u \in M_{n+1}(B)$ such that

$$\big\| u^* \operatorname{diag}\big(\phi(a), \sigma(a), \ldots, \sigma(a)\big) u - \operatorname{diag}\big(\psi(a), \sigma(a), \ldots, \sigma(a)\big) \big\| < \varepsilon$$

for all $a \in \mathcal{F}$, where $\sigma(a)$ is repeated n times.

Proof. Suppose that the theorem is false. Then there are $\varepsilon_0 > 0$ and a finite subset $\mathcal{F} \subset A$ such that there are a sequence of positive numbers $\{\delta_n\}$ with $\delta_n \downarrow 0$, an increasing sequence of finite subsets $\{\mathcal{G}_n\}$ whose union is dense in the unit ball of A, a sequence of finite subsets $\{\mathcal{P}_n\}$ of $\mathbf{P}(A)$ with $\overline{\bigcup_{n=1}^\infty \mathcal{P}_n} = \mathbf{P}(A)$ and with $U_n = U(A) \cap \mathcal{P}_n$, a sequence $\{k(n)\}$ of integers ($k(n) \nearrow \infty$) and sequences $\{\phi_n\}$, $\{\psi_n\}$ and $\{\sigma_n\}$ of $\mathcal{G}_n$-δ_n-multiplicative positive linear maps from A to B_n with $[\phi_n]|_{\mathcal{P}_n} = [\psi]|_{\mathcal{P}_n}$ and

$$\mathrm{cel}\big(\tilde{\phi}_n(u)^* \tilde{\psi}_n(u)\big) \leqslant \mathbf{L}(u)$$

for all $u \in U_n$ satisfying the following:

$$\inf\big\{\sup\big\{\big\| u^* \operatorname{diag}\big(\phi_n(a), \sigma_n(a), \ldots, \sigma_n(a)\big) u - \operatorname{diag}\big(\psi_n(a), \sigma_n(a), \ldots, \sigma_n(a)\big) \big\| \colon a \in \mathcal{F}\big\}\big\} \geqslant \varepsilon_0$$

where $\sigma_n(a)$ is repeated $k(n)$ times and the infimum is taken over all unitaries in $M_{k(n)+1}(B_n)$.

Set $D_0 = \bigoplus_{n=1}^\infty B_n$ and $D = \prod_{n=1}^\infty B_n$. Define Φ, Ψ, $\Sigma : A \to D$ by $\Phi(a) = \{\phi_n(a)\}$, $\Psi(a) = \{\psi_n(a)\}$ and $\Sigma(a) = \{\sigma_n(a)\}$ for $a \in A$. Let $\pi : D \to D/D_0$ be the quotient map and set $\bar{\Phi} = \pi \circ \Phi$, $\bar{\Psi} = \pi \circ \Psi$ and $\bar{\Sigma} = \pi \circ \Sigma$. Note that $\bar{\Phi}$, $\bar{\Psi}$ and $\bar{\Sigma}$ are monomorphisms. For any $u \in U_k$,

$$\mathrm{cel}\big(\tilde{\phi}_n(u)^* \tilde{\psi}_n(u)\big) \leqslant \mathbf{L}(u)$$

for all sufficiently large n ($> k$). This implies that there is an equi-continuous path $\{v_n(t)\}$ ($t \in [0,1]$) such that

$$v_n(0) = \tilde{\phi}_n(u) \quad \text{and} \quad v_n(1) = \tilde{\psi}_n(u)$$

(see, for example, [25, Theorem 1.1]). Therefore, we conclude that

$$[\bar{\Phi}]|_{K_1(A)} = [\bar{\Psi}]|_{K_1(A)}.$$

Given an element $p \in \mathcal{P}_k \setminus U_k$ (for some k), we claim that

$$\big[\bar{\Phi}(p)\big] = \big[\bar{\Psi}(p)\big].$$

We have (see [25, Proposition 2.1])

$$K_0\Big(\prod B_n\Big) = \prod^b K_0(B_n) \quad \text{and} \quad K_0(D/D_0) = \prod^b K_0(B_n)/\bigoplus K_0(B_n),$$

where $\prod^b K_0(B_n)$ is the sequences of elements $\{[p_n] - [q_n]\}$, where p_n and q_n can be represented by projections in $M_L(B_n)$ for some integer L. Since each $TR(B_n) \leqslant 1$, B_n has stable rank one and $K_0(B_n)$ is weakly unperforated. By [25, Proposition 2.2], each B_n has K_i-divisible rank T with $T(n,k) = 1$. By 6.5, $\mathrm{cer}(M_k(B_n)) \leqslant 4$ for all k and n, and the kernel of the map from $K_1(\prod_n B_n)$ to $\prod_n^b K_1(B_n)$ is divisible and torsion free (see 6.12). By the proof of [25, Theorem 2.1, part (2)], we also have

$$K_i\bigg(\prod_n B_n, \mathbf{Z}/m\mathbf{Z}\bigg) \subset \prod_n K_i(B_n, \mathbf{Z}/m\mathbf{Z}), \quad m = 2, 3, \ldots.$$

(In fact, by 6.10, each B_n has exponential length divisible rank E with $E(L,k) = 8/\pi + L/k + 1$ so that [25, Theorem 2.1(2)] can be applied directly. See also [25, Corollary 2.1, part (2)].)

Since $[\phi_n(p)] = [\psi_n(p)]$ in $K_0(B_n)$ or in $K_i(B_n, \mathbf{Z}/m\mathbf{Z})$ ($i = 0, 1$, $m = 2, 3, \ldots$) for large n,

$$\big[\bar{\Phi}(p)\big] = \big[\bar{\Psi}(p)\big].$$

Then $\bar{\Phi}_* = \bar{\Psi}_*$. Therefore $[\bar{\Phi}] = [\bar{\Psi}]$ in $KL(A, \prod_n B_n/\bigoplus_n B_n)$.

By applying 8.4, we obtain an integer N and a unitary $u \in M_{N+1}(D/D_0)$ such that

$$\big\|u^* \operatorname{diag}\big(\bar{\Phi}(a), \bar{\Sigma}(a), \ldots, \bar{\Sigma}(a)\big)u - \operatorname{diag}\big(\bar{\Psi}(a), \bar{\Sigma}(a), \ldots, \bar{\Sigma}(a)\big)\big\| < \varepsilon_0/3$$

for all $a \in \mathcal{F}$, where $\bar{\Sigma}(a)$ is repeated N times. It is easy to see (see [30, 1.3] for example) there is a unitary $U \in M_{N+1}(D)$ such that $\pi(U) = u$ and for each $a \in \mathcal{F}$ there exists $c_a \in M_{N+1}(D_0)$ such that

$$\big\|U^* \operatorname{diag}\big(\Phi(a), \Sigma(a), \ldots, \Sigma(a)\big)U - \operatorname{diag}\big(\Psi(a), \Sigma(a), \ldots, \Sigma(a)\big) + c_a\big\| < \varepsilon_0/3$$

where $\Sigma(a)$ is repeated N times. Write $U = \{u_n\}$, where $u_n \in M_{N+1}(B_n)$ are unitaries. Since $c_a \in M_{N+1}(D_0)$ and $\mathcal{F}$ is finite, there is $N_0 > 0$ such that for $n \geqslant N_0$

$$\left\|u_n^* \operatorname{diag}\bigl(\phi_n(a), \sigma_n(a), \ldots, \sigma_n(a)\bigr)u_n - \operatorname{diag}\bigl(\psi_n(a), \sigma_n(a), \ldots, \sigma_n(a)\bigr)\right\| < \varepsilon_0/2$$

for all $a \in \mathcal{F}$, where σ_n is repeated N times. This contradicts the assumption that the theorem is false. □

8.6. Theorem. *Let A be a separable unital nuclear simple C^*-algebra with $TR(A) \leqslant 1$ satisfying the AUCT and let $\mathbf{L} : U(A) \to \mathbf{R}_+$. Then for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta_1 > 0$, an integer $n > 0$, a finite subset $\mathcal{P} \subset \mathbf{P}(A)$, a finite subset $S \subset A$ satisfying the following:*

(i) *there exist mutually orthogonal projections $q, p_1, \ldots, p_n$ with $q \preccurlyeq p_1$ and $p_1, \ldots, p_n$ mutually unitarily equivalent, and there exists a C^*-subalgebra $C \in \mathcal{I}$ with $1_C = p_1$ and unital S-$\delta_1/2$-multiplicative contractive completely positive linear maps $\phi_0 : A \to qAq$ and $\phi_1 : A \to C$ such that*

$$\left\|x - \bigl(\phi_0(x) \oplus \bigl(\phi_1(x), \phi_1(x), \ldots, \phi_1(x)\bigr)\bigr)\right\| < \delta_1/2$$

for all $x \in S$, where $\phi_1(x)$ is repeated n times; moreover, there exist a finite subset $\mathcal{G}_0 \subset A$, a finite subset $\mathcal{P}_0$ of projections in $M_\infty(C)$, a finite subset $\mathcal{H} \subset A_{\mathrm{sa}}$, $\delta_0 > 0$ and $\sigma > 0$ (which depend on the choices of C);

for any unital simple C^*-algebra B with $TR(B) \leqslant 1$ and any two $\mathcal{S} \cup \mathcal{G}_0$-$\delta$-multiplicative completely positive linear contractions $L_1, L_2 : A \to B$ for which the following hold (with $\delta = \min\{\delta_1, \delta_0\}$):

(ii) $[L_1]|_{\mathcal{P}\cup\mathcal{P}_0} = [L_2]|_{\mathcal{P}\cup\mathcal{P}_0}$;
(iii) $|\tau \circ L_1(g) - \tau \circ L_2(g)| < \sigma$ *for all $g \in \mathcal{H}$ and $\tau \in T(A)$*;
(iv) $e = L_1 \circ \phi_0(1_A) = L_2 \circ \phi_0(1_A)$ *is a projection*;
(v) $\mathrm{cel}(\tilde{L}_1(\phi_0(u))^* \tilde{L}_2((\phi_0(u)))) \leqslant \mathbf{L}(u)$ *(in $U(eBe)$) for all $u \in U(A) \cap \mathcal{P}$*,

there exists a unitary $U \in B$ such that

$$\mathrm{ad}(U) \circ L_1 \approx_\varepsilon L_2 \quad \textit{on } \mathcal{F}.$$

Note that (i) holds as long as $TR(A) \leqslant 1$ and does not depend on $\mathbf{L}$, ε and $\mathcal{F}$.

Proof. Since $TR(A) \leqslant 1$, 8.5 applies. Fix a finite subset $\mathcal{F} \subset A$, $\varepsilon > 0$ and $\mathbf{L}$. Let $\delta_1 > 0$, $\mathcal{G}_1 \subset A$, $\mathcal{P} \subset \mathbf{P}(A)$ and n be as required by 8.5 corresponding to $\mathbf{L}$, $\varepsilon/4$ and $\mathcal{F}$.

Let $S = \mathcal{G}_1$, $\eta = \min(\delta_1/2, \varepsilon/4)$. Let $q, p_1, \ldots, p_n$, ϕ_0 and ϕ_1 satisfy (i) given by 5.5. Let $\delta_0' > 0$, $\sigma_1 > 0$, and a finite subset $\mathcal{G}_0' \subset A$ ($\mathcal{G}_2 \supset \phi_1(\mathcal{G}_1)$) be as required by 5.8 corresponding to C, the finite subset $\phi_1(S)$ and η. Let $\mathcal{P}_0$ contain a finite set of projections in C which generates $K_0(C)$.

Now choose a finite subset $\mathcal{G}_0'' \subset C$ and $\delta_2 > 0$ so that any $\mathcal{G}'$-δ_2-multiplicative contractive completely positive linear map L from C to any C^*-algebra gives a well defined map from $K_0(C)$.

Let $\mathcal{G}_0 = \mathcal{G}_0' \bigcup \mathcal{G}_0''$, $\delta_0 = \min(\delta_2, \delta_0)$ and $\sigma = \sigma_1/2(n+1)$, and let $\mathcal{H} = \{\frac{a^*+a}{2}, \frac{a-a^*}{2i} \colon a \in \mathcal{G}_0'\}$. Let $L_i : A \to B$ be two $\mathcal{S} \cup \mathcal{G}_0$-$\delta$-multiplicative contractive completely positive linear maps ($i = 1, 2$) which satisfy (ii)–(v). Note that $[L_1]|_{K_0(C)} = [L_2]|_{K_0(C)}$. Let $e_1 = \phi_1(1_A)$. By assumption,

e_1 is a projection in B. By the choice of δ_2, σ_1 and $\mathcal{G}_2$, applying 5.8, we obtain a unitary $v \in e_1Be_1$ such that

$$\|L_1(x) - v^*L_2(x)v\| < \eta$$

for all $x \in \phi_1(S)$. Therefore,

$$\|L_1\phi_1(a) - \mathrm{ad}(v) \circ L_2 \circ \phi_1(a)\| < \eta$$

for all $a \in \mathcal{G}_1$. To simplify notation, without loss of generality, we may assume that $L_2 \circ \phi_1 = \mathrm{ad}(v) \circ L_2 \circ \phi_1$.

Now, by applying 8.5, we have

$$L_1 \circ \phi_0 \oplus (L_1 \circ \phi_1, L_1 \circ \phi_1, \ldots, L_1 \circ \phi_1) \sim_{\varepsilon/4} L_2 \circ \phi_0 \oplus (L_1 \circ \phi_1, L_1 \circ \phi_1, \ldots, L_1 \circ \phi_1)$$

on $\mathcal{F}$. From the above ($\eta < \varepsilon/4$), we obtain

$$L_2 \circ \phi_0 \oplus (L_1 \circ \phi_1, L_1 \circ \phi_1, \ldots, L_1 \circ \phi_1) \sim_{\varepsilon/2} L_2 \circ \phi_0 \oplus (L_2 \circ \phi_1, \ldots, L_2 \circ \phi_1)$$

on $\mathcal{F}$. Therefore

$$L_1 \sim_{\varepsilon} L_2 \quad \text{on } \mathcal{F}. \qquad \square$$

It turns out that when $K_1(A)$ is torsion the "uniqueness theorem" can be stated in a much more easy way.

8.7. Theorem. *Let A be a unital separable simple C^*-algebra with $TR(A) \leqslant 1$ and with torsion $K_1(A)$. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$ there exist $\delta > 0$, $\sigma > 0$, a finite subset $\mathcal{P} \subset \mathbf{P}(A)$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: for any unital simple C^*-algebra B with $TR(B) \leqslant 1$, any two $\mathcal{G}$-δ-multiplicative completely positive linear contractions $L_1, L_2 : A \to B$ with*

$$[L_1]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}}$$

and

$$\sup_{\tau \in T(B)} \{|\tau \circ L_1(g) - \tau \circ L_2(g)|\} < \sigma$$

for all $g \in \mathcal{G}$, there exists a unitary $U \in B$ such that

$$\mathrm{ad}(U) \circ L_1 \approx_{\varepsilon} L_2 \quad \text{on } \mathcal{F}.$$

Proof. Define $\mathbf{L} : U(A) \to \mathbf{R}_+$ as follows. If $u \in U_0(A)$, we let $\mathbf{L}(u) = 2(\mathrm{cel}(u) + \pi/16)$; if $u \in U(A) \setminus U_0(A)$, but $[u]$ has order $k > 1$ in $K_1(A)$ (therefore $u^k \in U_0(A)$ since A has stable rank one), let $\mathbf{L}(u) = 8\pi + \frac{2\,\mathrm{cel}(u^k)}{k} + \pi/16$.

Let $\mathcal{G}_1 = \mathcal{G}(\mathcal{F}, \varepsilon/2)$, $\delta_1 = \delta(\mathcal{F}, \varepsilon/2)$, $n = n(\mathcal{F}, \varepsilon/2)$, $S_1 = S(\mathcal{F}, \varepsilon/2)$ and $\mathcal{P}_1 = \mathcal{P}(\mathcal{F}, \varepsilon/2)$ be as required in Theorem 8.6 (corresponding to $\mathcal{F}$, $\varepsilon/2$ and $\mathbf{L}$). Choose a finite subset $\mathcal{G}_2$ and a positive number $\eta < \min(\varepsilon/2, \delta_1/2)$ satisfying the following: if $H_i : A \to B$ are both $\mathcal{G}_2$-η-multiplicative with

$$H_1 \approx_\eta H_2 \quad \text{on } \mathcal{G}_2$$

then $[H_1]|_{\mathcal{P}_1} = [H_2]|_{\mathcal{P}_1}$.

If $u \in U(A)$, denote by $k(u)$ the smallest integer for which $[u^{k(u)}] \in U_0(A)$. Let $K = \max\{k(u)\colon\ u \in \mathcal{P}_1 \cap U(A) \setminus U_0(A)\}$. Let $\mathcal{P}_2 = \mathcal{P}_1 \cup \{u^{k(u)}\colon\ u \in U(A) \cap \mathcal{P}_1\}$. Set $V = \{v^1, v^2, \ldots, v^K, v \in \mathcal{P}_1 \cap (U(A) \setminus U_0(A))\}$. We may assume that $\mathcal{G}_2 \supset V \cup \mathcal{G}_1 \cup S_1$. Set $0 < \delta_2 < \min(\delta_1/2, \eta/4, \varepsilon/4, 1/K^2 512)$.

Since A is a nuclear simple C^*-algebra with $TR(A) \leqslant 1$, from Lemma 5.5, there exist a C^*-subalgebra $F \in \mathcal{I}$ of A with $p_1 = 1_F$ and $\mathcal{G}_2$-δ_2-multiplicative completely positive linear contraction $\phi_1 : A \to F$ such that

$$\mathrm{id}_A(x) \approx_{\delta_2} qxq \oplus \mathrm{diag}\big(\phi_1(x), \phi_1(x), \ldots, \phi_1(x)\big) \quad \text{for all } x \in \mathcal{G}_2,$$

where ϕ_1 is repeated n times and $(1-q)A(1-q) = M_n(p_1Ap_1)$. Set $\phi_0(x) = qxq$ for $x \in A$. Now let $\mathcal{P}_0$, $\mathcal{G}_0$, $\delta_0 > 0$, $\sigma > 0$ and finite subset $\mathcal{H} \subset A_{\mathrm{sa}}$ be as required in of 8.6(i).

By 6.8 (with perhaps larger $\mathcal{G}_2$), we also have (in $U(qAq)$)

$$\mathrm{cel}\big(\tilde{\phi}_0(u)\big) \leqslant \mathrm{cel}(u) + \pi/128$$

for $u \in U_0(A) \cap \mathcal{P}_1$; and

$$\mathrm{cel}\big(\tilde{\phi}_0\big(u^k\big)\big) \leqslant \mathrm{cel}\big(u^k\big) + \pi/128 \quad \text{and}$$
$$\big\|\tilde{\phi}_0\big(u^k\big) - \tilde{\phi}_0(u)^k\big\| < 1/128$$

for $u \in (U(A) \setminus U_0(A)) \cap \mathcal{P}_1$ and $[u]$ has order k in $K_1(A)$.

Let $\mathcal{G} \supset \mathcal{G}_2 \cup \phi_0(\mathcal{G}_2) \cup \mathcal{G}_0$ be a finite subset and $0 < \delta < \delta_2$. Let $\mathcal{P} = \mathcal{P}_1 \cup \{q, p_1\}$. Suppose that L_i are two unital $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps that satisfy

$$[L_1]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}} \quad \text{and} \quad \sup_{\tau \in T(B)} \big\{\big|\tau \circ L_1(g) - \tau \circ L_2(g)\big|\big\} < \sigma/2$$

for all $g \in \mathcal{H}$.

For any (finite) $\mathcal{G}_3 \subset \mathcal{G}_2 \cup \phi_0(\mathcal{G}_2)$ and $0 < \delta_3 < \delta_2/2$, by considering $(L_i)|_{qBq \oplus (1-q)B(1-q)}$, from 8.2(iii), with possibly larger $\mathcal{G}$ and smaller δ, there are $\mathcal{G}_3$-δ_3-multiplicative contractive completely positive linear maps $L_i' : A \to B$ $(i = 1, 2)$ such that

$$\big\|L_i(g) - L_i'(g)\big\| < \delta_3 \quad \text{for all } g \in \mathcal{G}_3$$

and $L_i'(q)$ is a projection $(i = 1, 2)$. Furthermore, we may assume that

$$\big[L_1'\big]|_{\mathcal{P}} = \big[L_2'\big]|_{\mathcal{P}} \quad \text{and} \quad \big|\tau \circ L_1'(g) - \tau \circ L_2'(g)\big| < \sigma$$

for all $g \in \mathcal{H}$ and $\tau \in T(B)$. Replacing L_2' by $\mathrm{ad}(W) \circ L_2'$ for a suitable unitary $W \in B$, we may assume that $e = L_1'(q) = L_1'(\phi_0(1_A)) = L_2'(q)$.

Let $\Lambda_i = L_i' \circ \phi_0$. With sufficiently large $\mathcal{G}_3$ and sufficiently small δ_3 (applying 6.8), we may assume that

$$\mathrm{cel}\big(\widetilde{\Lambda}_i(u)\big) \leqslant \mathrm{cel}(u) + \pi/64,$$

$i = 1, 2$, $u \in U_0(A) \cap \mathcal{P}_1$;

$$\big\| \widetilde{\Lambda}_i\big(u^k\big) - \tilde{\Lambda}_i(u)^k \big\| < 1/64 \quad \text{and} \quad \mathrm{cel}\big(\widetilde{\Lambda}_i\big(u^k\big)\big) \leqslant \mathrm{cel}\big(u^k\big) + \pi/64$$

for $u \in (U(A) \setminus U_0(A)) \cap \mathcal{P}_1$ but $[u]$ has order k in $K_1(A)$. Then, for $u \in U_0(A) \cap \mathcal{P}_1$

$$\mathrm{cel}\big(\tilde{L}_1'\big(\phi_0(u)\big)^* \tilde{L}_2'\big(\phi_0\big((u)\big)\big)\big) \leqslant 2\big(\mathrm{cel}(u) + \pi/64\big) \leqslant \mathbf{L}(u);$$

and, for $u \in (U(A) \setminus U_0(A)) \cap \mathcal{P}_1$ with order k, since

$$\mathrm{cel}\big(\tilde{L}_1'\big(\phi_0\big(u^k\big)\big)^* \tilde{L}_2'\big(\phi_0\big(u^k\big)\big)\big) < 2\big[\mathrm{cel}\big(u^k\big) + \pi/64\big],$$

by 6.10,

$$\mathrm{cel}\big(\tilde{L}_1'\big(\phi_0(u)\big)^* \tilde{L}_2'\big(\phi_0(u)\big)\big) \leqslant 8\pi + \frac{2[\mathrm{cel}(u^k) + \pi/64]}{k} + \pi/64 < \mathbf{L}(u).$$

Now Theorem 8.6 provides a unitary $U \in B$ such that

$$\mathrm{ad}(U) \circ L_2' \approx_{\varepsilon/2} L_1' \quad \text{on } \mathcal{F}.$$

This implies that

$$L_2 \sim_{\varepsilon} L_1 \quad \text{on } \mathcal{F}. \qquad \square$$

The following is a characterization of approximately inner automorphisms (for the case in which $K_1(A)$ is torsion).

8.8. Theorem. *Let A be a unital nuclear simple C^*-algebra with $TR(A) \leqslant 1$ and with torsion $K_1(A)$ which satisfies the AUCT. Then an automorphism $\alpha : A \to A$ is approximately inner if and only if $[\alpha] = [\mathrm{id}_A]$ in $KL(A, A)$ and $\tau \circ \alpha(x) = \tau(x)$ for all $x \in A$ and $\tau \in T(A)$.*

Proof. If α is approximately inner, then it is clear that

$$\tau \circ \alpha(x) = \tau(x)$$

for all $x \in A$ and $\tau \in T(A)$. The "only if" part follows from [35, 4.5]. It is also clear that the "if part" follows from 8.7. $\square$

9. The existence theorems

9.1. Definition. Let A and B be two unital stably finite C^*-algebras and let $\alpha : K_0(A) \to K_0(B)$ be a positive homomorphism and $\Lambda : T(B) \to T(A)$ be a continuous affine map. We say Λ is compatible with α if $\Lambda(\tau)(x) = \tau(\alpha(x))$ for all $x \in K_0(A)$, where we view τ as a state on $K_0(A)$. Let S be a compact convex set. Denote by $\mathit{Aff}(S)$ the set of all (real) continuous affine functions on S. Let $\Lambda : S \to T$ be a continuous affine map from S to another compact convex set T. We denote by $\Lambda_\natural : \mathit{Aff}(T) \to \mathit{Aff}(S)$ the unital positive linear continuous map defined by $\Lambda_\natural(f)(s) = f(\Lambda(s))$ for $f \in \mathit{Aff}(T)$. A positive linear map $\xi : \mathit{Aff}\, T(A) \to \mathit{Aff}\, T(B)$ is said to be compatible with α if $\xi(\hat{p})(\tau) = \tau(\alpha(p))$ for all $\tau \in T(B)$ and any projection $p \in M_\infty(A)$. Let A be a unital C^*-algebra (with at least one normalized trace). Define $Q : A_{\mathrm{sa}} \to \mathit{Aff}\, T(A)$ by $Q(a)(\tau) = \tau(a)$ for $a \in A$. Then Q is a unital positive linear map.

A C^*-algebra A is said to be *KK-attainable* for a class of stably finite C^*-algebras $\mathcal{C}$, if for any C^*-algebra $B \in \mathcal{C}$, any $\alpha \in \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))^{++}$ (see 8.2) and any finite subset $\mathcal{P} \subset \mathbf{P}(A)$ with $[1_A] \subset \mathcal{P}$, there exists a sequence of completely positive linear contractions $L_n : A \to B \otimes \mathcal{K}$ such that

$$\big\| L_n(a)L_n(b) - L_n(ab)\big\| \to 0 \quad \text{and} \quad [L_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}} \quad \text{for all } a, b \in A.$$

For the rest of the paper, when we say a C^*-algebra A is *KK*-attainable, we mean that A is *KK*-attainable for unital separable simple C^*-algebras with tracial rank no more than 1.

As in [32], if for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists a C^*-subalgebra A_1 of A which is *KK*-attainable such that $\mathcal{F} \subset_\varepsilon A_1$, then A is *KK*-attainable.

A unital nuclear separable simple C^*-algebra A with $\mathit{TR}(A) \leqslant 1$ is said to be *pre-classifiable* if it satisfies the Universal Coefficient Theorem and is *KK*-attainable, and, in addition to the above, for any unital separable nuclear simple C^*-algebra with $\mathit{TR}(A) \leqslant 1$ and any continuous affine map $\Lambda : T(B) \to T(A)$ compatible with α,

$$\sup_{\tau \in T(B)} \big\{\big|\Lambda(\tau)(a) - \tau \circ L_n(a)\big|\big\} \to 0 \quad \text{for all } a \in A.$$

Or, equivalently, for any contractive positive linear map $\xi : \mathit{Aff}\, T(A) \to \mathit{Aff}\, T(B)$ compatible with α,

$$\sup_{\tau \in T(B)} \big\{\big|\xi\big(Q(a)\big)(\tau) - \tau \circ L_n(a)\big|\big\} \to 0 \quad \text{for all } a \in A_{\mathrm{sa}}.$$

9.2. If $h : A \to B$ is a unital homomorphism, then h induces a unital positive affine map $h_\sharp : \mathit{Aff}\, T(A) \to \mathit{Aff}\, T(B)$. The map $h_\sharp$ is contractive. Suppose that Y is a compact metric space and $P \in M_l(C(Y))$ is a non-zero projection with constant rank. It is known and easy to see that

$$\mathit{Aff}\, T\big(PM_l\big(C(Y)\big)P\big) = \mathit{Aff}\, T(M_l\big(C(Y)\big) = C_{\mathbf{R}}(Y).$$

9.3. Theorem. *Let A be a simple unital C^*-algebra with at least one tracial state. Then for any affine function $f \in \mathit{Aff}(T(A))$ with $\|f\| \leqslant 1$ and any $\varepsilon > 0$, there exists an element $a \in A_{\mathrm{sa}}$ with $\|a\| < \|f\| + \varepsilon$ such that $\tau(a) = f(\tau)$ for all $\tau \in T(A)$. Furthermore, if $f > 0$, we can choose $a \geqslant 0$.*

Proof. We prove this using the results in [8]. By [8, 2.7], we may identify $T(A)$ with the real part of the unit sphere of $(A^q)^*$ (see [8] for the notation). By [8, 2.8], it suffices to consider those $f \in \mathit{Aff}(T(A))$ with $f(\tau) > 0$ for all $\tau \in T(A)$. There is an element $b \in (A^q)^{**}$ such that $b(\tau) = f(\tau)$ for all $\tau \in T(A)$. Since f is (weak-*) continuous, $b \in A^q$. Since $b(\tau) > 0$ for all $\tau \in T(A)$, by [8, 6.4], there is $c \in A_+$ and $z \in A_{sa}$ with $\tau(z) = 0$ for all $\tau \in T(A)$ (i.e., $z \in A_0$ using the notation in [8]) such that $b = c + z$. Now the theorem follows from [8, 2.9]. □

9.4. Lemma. *Let A be a separable unital C^*-algebra. Let $\varepsilon > 0$ and $\mathcal{F} \subset A$ be a finite subset. Then there exists $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: for any unital separable C^*-algebra C with at least one tracial state and any unital contractive positive linear maps $L: A \to C$ which is $\mathcal{G}$-δ-multiplicative, then, for any $t \in T(C)$ there is a trace $\tau \in T(A)$ such that*

$$\left|\tau(a) - t\big(L(a)\big)\right| < \varepsilon \quad \textit{for all } a \in \mathcal{F}.$$

Proof. Otherwise, there would be an $\varepsilon_0 > 0$ and a finite subset $\mathcal{F} \subset A$, a sequence of unital separable C^*-algebra C_n, a sequence of unital contractive positive linear map $L_n : A \to C_n$ such that

$$\lim_{n\to\infty} \left\| L_n(a)L_n(b) - L_n(ab)\right\| = 0 \quad \text{for all } a, b \in A,$$

and a sequence $t_n \in T(C_n)$ such that

$$\inf\left\{\max\left\{\left|t(a) - t_n\big(L_n(a)\big)\right|: a \in \mathcal{F}\right\}: t \in T(A)\right\} \geqslant \varepsilon_0$$

for all n. Let s_n be a state of A which extends $t_n \circ L_n$. Let τ be a weak limit of $\{s_n\}$. So there is a subsequence $\{n_k\}$ such that $\tau(a) = \lim_{m\to\infty} s_{n_k}(a)$ for all $a \in A$. It is a routine to check that $\tau \in T(A)$. Therefore, there exists $K > 0$, such that

$$\left|\tau(a) - t_{n_k}\big(L_{n_k}(a)\big)\right| < \varepsilon_0/2$$

for all $k \geqslant K$. We obtain a contradiction. □

9.5. Lemma. *Let $A = C(X)$, where X is a path connected finite CW-complex. Let B be a unital separable nuclear non-elementary simple C^*-algebra with $TR(B) \leqslant 1$ and $\Lambda : T(B) \to T(A)$ be a continuous affine map. Then, for any $\sigma > 0$ and any finite subset $\mathcal{H} \subset \mathit{Aff}\,T(A)$, there exists a unital monomorphism $h : A \to B$ such that the image of h is in a C^*-subalgebra $B_0 \in \mathcal{I}$ and*

$$\left\|h_\sharp(f) - \Lambda_\natural(f)\right\| < \varepsilon$$

for all $f \in \mathcal{H}$, where $h_\sharp, \Lambda_\natural : \mathit{Aff}\,T(A) \to \mathit{Aff}\,T(B)$ are the maps induced by h and Λ, respectively.

Moreover, if there is positive homomorphism $\alpha : K_0(A) \to K_0(B)$ with $\alpha([1_A]) = [1_B]$ and $\Lambda_\natural$ is compatible with α, then the above is also true for $A = PM_l(C(X))P$, where $P \in M_l(C(X))$ is a projection in $M_l(C(X))$. Furthermore, if X is contractible, we can also require that $h_{*0} = \alpha$.

Proof. We will apply [27, 2.5]. We may assume that the identity of A is contained in $\mathcal{H}$. Fix $\varepsilon > 0$. By 9.3, for each $f \in \Lambda_\natural(\mathcal{H})$, there is $a_f \in B_{\mathrm{sa}}$ with $\|a_f\| \leqslant \|f\|$ such that

$$\sup_{\tau \in T(B)} \left\{ \left| \tau(a_f) - \tau(f) \right| \right\} < \varepsilon/32.$$

Let $\mathcal{G} = \{a_f\colon f \in \Lambda_\natural(\mathcal{H})\}$. Let $N(2)$ be the number described in [27, 2.5] corresponding to $\varepsilon/4$ and $\mathcal{H}$.

Since $TR(B) \leqslant 1$, by 4.10, for any $\delta > 0$ and any finite subset $\mathcal{G}_1 \subset A$, (we assume that $\mathcal{G} \subset \mathcal{G}_1$), there exist a nonzero projection $p \in B$ and a unital C^*-subalgebra $C \in \mathcal{I}$ with $1_C = p$ such that:

(1) $\|[x, p]\| < \delta$ for all $x \in \mathcal{G}_1$,
(2) $pxp \in_\delta C$ for all $x \in \mathcal{G}_1$,
(3) $\tau(1 - p) < \varepsilon/64$, and
(4) $C = \bigoplus_{i=1}^n C_i$ with $C_i = M_{m(i)}(C([0, 1]))$ and $m(i) \geqslant N(2)$.

We have

$$\left| \tau(pxp) - \tau(x) \right| < \varepsilon/32 \quad \text{for all } \tau \in T(B) \text{ and for all } x \in \mathcal{G}.$$

Since B is nuclear, by [32, 3.2], there exists a unital completely positive contraction $L' : pBp \to C$ such that

$$\left\| L'(pxp) - pxp \right\| < \varepsilon/32 \quad \text{for all } x \in \mathcal{G}.$$

Define $L : B \to C$ by $L(b) = L'(pbp)$ for $b \in B$. With sufficiently small δ, we have

$$\sup_{\tau \in T(B)} \left\{ \left| t \circ L(x) - \tau(x) \right|\colon \tau \in T(B) \quad \text{and} \quad t = \tau/\tau(p) \right\} < \varepsilon/16$$

for all $x \in \mathcal{G}$. Choose an integer $N > 0$ so that

$$\left| g(t') - g(t') \right| < \varepsilon/32 \quad \text{if } |t' - t''| \leqslant 1/N \text{ for all } g \in \Lambda_\natural(\mathcal{H}).$$

Here we view $\mathit{Aff}(C) = \bigoplus_{i=1}^n C_{\mathbf{R}}([0, 1])$ and t' and t' are points in the same $[0, 1]$. Let $0 = t_{i,0} < t_{i,1} < \cdots < t_{i,N} = 1$ such that $t_{i,k} - t_{i,k+1} = 1/N$ and i indicates the ith interval. We also view them as tracial states of C, i.e., $t_{i,k}(f) = \mathrm{tr}(f(t_{i,k}))$ for $f \in \bigoplus_{i=1}^n C([0, 1], M_{m(i)})$, where tr is the normalized trace on $M_{m(i)}$.

By applying Lemma 9.4, with sufficiently large $\mathcal{G}_1$ and small δ, we may assume that there are $\tau_{i,k} \in T(B)$ such that

$$\left| \tau_{i,k}(a) - t_{i,k}\bigl(L(a)\bigr) \right| < \varepsilon/32$$

for all $a \in \mathcal{G}$. Let Δ be the convex hull of $t_{i,k}$ in $T(C)$. Define $\gamma_1 : \Delta \to T(B)$ by $\gamma_1(\sum_{i,k} \alpha_{i,k} t_{i,k}) = \sum_{i,k} \alpha_{i,k} \tau_{i,k}$ for $\alpha_{i,k} \geqslant 0$ and $\sum_{i,k} \alpha_{i,k} = 1$. Therefore

$$\left| \gamma_1(t)(a) - t\bigl(L(a)\bigr) \right| < \varepsilon/32 \quad \text{for all } a \in \mathcal{G}$$

and $t \in \Delta$. For each $\tau \in T(C)$, one has, for $f \in \bigoplus_{i=1}^{n} C([0,1], M_{m(i)})$, that

$$\tau(f) = \sum_{i=1}^{n} \int \operatorname{tr}\big(f(t)\big) d\mu_i,$$

where μ_i is a Borel measure on $[0,1]$. Define $\gamma_2 : T(C) \to \Delta$ by

$$\gamma_2(\tau)(f) = \sum_{i=1}^{n} \sum_{k=0}^{N} \alpha_{i,k} \operatorname{tr}\big(f(t_{i,k})\big)$$

for $f \in C$, where $\alpha_{i,k} = \mu_i(A_k)$ and $A_k = [t_{i,k}, t_{i,k+1})$, $k = 0, 1, 2, \ldots, N$. Put $\gamma = \gamma_1 \circ \gamma_2$. It is clear that γ is an affine continuous map from $T(C)$ into $T(B)$. Moreover,

$$\big|\gamma(t)(a) - t\big(L(a)\big)\big| < \varepsilon/16$$

for all $f \in \mathcal{G}$ and $t \in T(C)$.

By [27, 2.5], there is a unital homomorphism $h_1 : A \to C$ such that

$$\big\|(h_1)_\sharp(g) - \gamma_\natural \circ \Lambda_\natural(g)\big\| < \varepsilon/16$$

for all $g \in \mathcal{H}$. In the following estimation, if $\tau \in T(B)$, we denote $\bar{\tau} = (1/\tau(p))\tau|_C$, where C is regarded as a subalgebra of A. We also note that $\Lambda_\natural(g)(\gamma(\bar{\tau})) = \gamma(\bar{\tau})(a_f)$ for $f = \Lambda_\natural(g)$ and $g \in \mathit{Aff}(T(A))$. We estimate, that, for $g \in \mathcal{H}$, $f = \Lambda_\natural(g)$, and for any $\tau \in T(B)$,

$$\begin{aligned}\big|(h_1)_\sharp(g)(\bar{\tau}) - \Lambda_\natural(g)(\tau)\big| &\leqslant \big|(h_1)_\sharp(g)(\bar{\tau}) - \gamma_\natural \circ \Lambda_\natural(g)(\bar{\tau})\big| \\ &\quad + \big|\gamma(\bar{\tau})(a_f) - \bar{\tau}\big(L(a_f)\big)\big| + \big|\bar{\tau}(L(a_f) - \Lambda_\natural(g)(\tau)\big|\end{aligned}$$

for $g \in \mathcal{H}$. Each of the first two terms on the right-hand side of the inequality is no more than $\varepsilon/16$. For the last term, we note that

$$\Lambda_\natural(g)(\tau) = \tau(a_f) \quad \text{and} \quad \big|\bar{\tau}\big(L(a_f)\big) - \tau(a_f)\big| < \varepsilon/16$$

for all $g \in \mathcal{H}$. Thus we have

$$\big|(h_1)_\sharp(g)(\bar{\tau}) - \Lambda_\natural(g)(\tau)\big| < 3\varepsilon/16 \quad \text{for all } g \in \mathcal{H}.$$

Since $(1-p)B(1-p)$ is non-elementary and simple, by [1, p. 61], there exists a positive element $b \in (1-p)B(1-p)$ with $\mathrm{sp}(b) = [0,1]$. From this we know that there is a unital C^*-subalgebra C_0 of $(1-p)B(1-p)$ such that $C_0 \cong C([0,1])$. It is well known that there is a unital monomorphism $h_2 : A \to C_0$. Finally, we let $h = h_1 + h_2$. It is clear that h meets the requirements of the conclusion of the (first part of) lemma.

For the second part, we note there is an integer $N > 0$ and a projection $e \in M_N(A)$ such that $eM_N(A)e \cong C(X)$. Let $d \in B$ be a projection such that $\alpha([e]) = [d]$. Since $\Lambda_\natural$ is compatible with α, $\Lambda_\natural$ induces a unital positive map $\zeta : \mathit{Aff}\, T(eAe) \to \mathit{Aff}\, T(dBd)$. From what we have proved, we obtain a homomorphism $h_1 : eAe \to dBd$ as required (with $\delta \cdot \varepsilon$, where $\delta = \inf\{\tau(d) :$

$\tau \in T(B)\}$). Set $h_2 = h_1 \otimes \mathrm{id}_{M_l} : eAe \otimes M_l \to dAd \otimes M_l$. Since $eAe \cong C(X)$, we may assume that $P \in eAe \otimes M_l$. Let $p \in dAd \otimes M_l$ such that $p = h_2(P)$. Since $\Lambda_\natural$ is compatible with α, $[p] = [1_B]$. Since B has stable rank one, p is unitarily equivalent to 1_B. Therefore $p(dAd \otimes M_l)p \cong A$. The second part of the lemma follows.

For the last part of the lemma, we note that $PM_l(C(X))P = M_s(C(X))$ for some integer $0 < s \leqslant l$, since X is contractible. Let e_{11} be a minimal projection of A. Choose a projection $d \in B$. For any finite subset $\mathcal{H}_1 \subset (e_{11}Ae_{11})_{\mathrm{sa}}$, from what we have shown, we obtain a homomorphism $h' : e_{11}Ae_{11} \to dBd$ such that

$$\left\| h'_\sharp(f) - \Lambda_\natural(f) \right\| < \varepsilon/s.$$

View $M_r(dBd)$ as a unital hereditary C^*-subalgebra of B. Put $h = h' \otimes \mathrm{id}_{M_r}$. It is clear that h meets the requirements of the lemma. □

9.6. Corollary. *Let $A \in \mathcal{I}$, B be a unital separable nuclear simple C^*-algebra with $TR(B) \leqslant 1$, $\gamma : K_0(A) \to K_0(B)$ be a positive homomorphism and $\Lambda : T(B) \to T(A)$ be an affine continuous map which is compatible with γ. Then, for any $\sigma > 0$ and any finite subset $\mathcal{G} \subset A$, there exists a unital monomorphism $\phi : A \to B$ such that*

$$\sup_{\tau \in T(B)} \left\{ \left| \tau \circ \phi(g) - \Lambda(\tau)(g) \right| \right\} < \sigma$$

for all $g \in \mathcal{G}$ and $\phi_* = \gamma$.

Proof. Note that 9.5 holds for $A = M_n$. It is then clear that, by considering each summand of A, the corollary follows from 9.5. □

9.7. Proposition. *Every KK-attainable, unital separable nuclear simple C^*-algebra A with $TR(A) \leqslant 1$ which satisfies the AUCT is pre-classifiable.*

Proof. Let A be a KK-attainable separable nuclear simple C^*-algebra with $TR(A) \leqslant 1$ satisfying the AUCT and B be a unital nuclear separable simple C^*-algebra with $TR(B) \leqslant 1$.

Let $\alpha \in \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))^{++}$, $\mathcal{P} \subset \mathbf{P}(A)$ be a finite subset containing $[1_A]$, and $\Lambda : T(B) \to T(A)$ be a continuous map which is comparable to $\alpha|_{K_0(A)}$. Suppose that $e \in B$ is a projection such that $\alpha(1_A) = e$. To save notation, without loss of generality, we may assume that $B = e(B \otimes \mathcal{K})e$. Let $\{\delta_n\}$ be a decreasing sequence of positive numbers with $\lim_{n\to\infty} \delta_n = 0$. For each n, since A is a unital simple C^*-algebra with $TR(A) \leqslant 1$, there are nonzero projections $p_n \in A$ and a C^*-subalgebra $C_n \in \mathcal{I}$ with $1_{C_n} = p_n$, and a sequence of unital completely positive linear contractions $\Phi_n : A \to C_n$ such that:

(1) $\|[x, p_n]\| < \delta_n$,
(2) $\|p_n x p_n - \Phi_n(x)\| < \delta_n$
(3) $\|x - (p_n x p_n \oplus \Phi_n(x))\| < \delta_n$ for all $x \in A$ with $\|x\| \leqslant 1$ and
(4) $\tau(1 - p_n) < 1/2n$ for all $\tau \in T(A)$.

Denote by $\Psi_n(x) = (1 - p_n)x(1 - p_n) + \Phi_n(x)$ (for $x \in A$). Note that

$$\|\Psi_n(ab)-\Psi_n(a)\Psi_n(b)\|\to 0 \quad\text{and}\quad \|\Phi_n(ab)-\Phi_n(a)\Phi_n(b)\|\to 0$$

for all $a,b\in A$ as $n\to\infty$.

Since A is KK-attainable, for each n, there exists a sequence of completely positive linear contractions $L_n : A\to B\otimes\mathcal{K}$ such that

$$[\Psi_n]|_{\mathcal{P}}=[\mathrm{id}]|_{\mathcal{P}},\qquad [L_n]|_{\mathcal{P}}=\alpha|_{\mathcal{P}},\qquad [L_n\circ\Psi_n]|_{\mathcal{P}}=\alpha|_{\mathcal{P}},$$
$$\|L_n\circ\Psi_n(ab)-L_n\circ\Psi_n(a)L_n\circ\Psi_n(b)\|\to 0\quad\text{and}$$
$$\|L_n\circ\Phi_n(ab)-L_n\circ\Phi_n(a)L_n\circ\Phi_n(b)\|\to 0$$

as $n\to\infty$ for all $a,b\in A$. Suppose that $C_n=\bigoplus_{i=1}^{t(n)}D_{n,i}$, where $D_{n,i}\cong M_{(n,i)}$ or $D_{n,i}\cong M_{(n,i)}(C([0,1]))$. Let $d_{n,i}=\mathrm{id}_{D_{n,i}}$. We may also assume that, for each n, $L_n(d_{n,i})$ is a projection (see 8.2(iii)) and

$$[L_n]\big([d_{n,i}]\big)=\alpha\big([d_{n,i}]\big)\quad\text{for all } n,i.$$

Let $\gamma_n : T(A)\to T(C_n)$ be defined by $\gamma_n(\tau)=(1/\tau(p_n))\tau|_{C_n}$. Let $\mathcal{G}_n$ be a finite subset (containing generators) of C_n and let $\{d_n\}$ be a decreasing sequence of positive numbers with $\lim_{n\to\infty}d_n=0$. For large n, by applying 9.6, we obtain a homomorphism $h_n : C_n\to q_nBq_n$, where $[d_{n,i}]=\alpha([d_{n,i}])$, such that

$$\big|\tau\circ h_n(g)-\gamma_n\circ\Lambda(\tau)(g)\big|<d_n\quad\text{for all } \tau\in T(B) \text{ and}$$

for all $g\in\mathcal{G}_n$. Put $\phi_n(x)=L_n\circ((1-p_n)x(1-p_n))+h_n\circ\Phi_n(x)$ for $x\in A$. It is easy to see, by choosing a large n, $\phi_n : A\to B$ meets the requirements of Definition 9.1. $\square$

9.8. Lemma. *Let A be a unital C^*-algebra, B be a unital separable simple C^*-algebra with $TR(B)\leqslant 1$ and $F\in\mathcal{I}$ be a C^*-subalgebra of B. Let G be a subgroup generated by a finite subset of $\mathbf{P}(A)$. Suppose that there is an $\mathcal{F}$-δ -multiplicative contractive completely positive linear map $\psi : A\to F\subset B$ such that $[\psi]|_G$ is well defined. Then for any $\varepsilon>0$, there exists a finite-dimensional C^*-subalgebra $C\subset B$ and an $\mathcal{F}$-δ–multiplicative contractive completely positive linear map $L : A\to C\subset B$ such that*

$$[L]|_{G\cap K_0(A,\mathbf{Z}/k\mathbf{Z})}=[\psi]|_{G\cap K_0(A,\mathbf{Z}/k\mathbf{Z})},\quad \text{and}\quad \tau(1_C)<\varepsilon$$

for all tracial states τ in $T(B)$ and for all $k\geqslant 1$ so that $G\cap K_0(A,\mathbf{Z}/k\mathbf{Z})\neq\{0\}$, where L and ψ are viewed as maps to B. Furthermore, if $[\psi]_{G\cap K_0(A)}$ is positive, so is $[L]|_{G\cap K_0(A)}$.

Proof. This is a minor modification of the proof of [32, 4.2]. Let $0<\varepsilon<1$. Without loss of generality, we may assume that $F=C([0,1])\otimes M_n$. Let $q_1\in F$ be a minimal projection. Suppose that

$$G\cap K_0(A,\mathbf{Z}/k\mathbf{Z})=\{0\}\quad\text{for } k>K.$$

By 5.5, with $m=2lK!+1$ and $1/l<\varepsilon/n$, we may write $q_1=q+\sum_{i=1}^m p_i$, where $[q]\leqslant[p_1]$, $q,p_1,\dots,p_m$ are mutually orthogonal projections, $[p_1]=[p_i]$, $i=1,2,\dots,m$ and $\tau(p_1)<$

$1/2l < \varepsilon/2n$. Set $e_1 = q + p_1$ and $q_0 = \sum_{j=2}^{2l+1} p_j$. Then $[e_1] + K![q_0] = [q_1]$ in $K_0(B)$ and $\tau(e_1) < \varepsilon/n$ for all tracial states τ on B. From this we obtain a C^*-subalgebra C of B such that $C \cong M_n$ and its minimal projection is equivalent to e_1. In particular, $\tau(1_C) < \varepsilon$. Let $\phi : F \to M_n \to C$ be a unital homomorphism, where the map $F \to M_n$ is a point-evaluation. Let $L = \phi \circ \psi$, $j_1 : F \to B$ and $j_2 : C \to B$ be embeddings. By the choice of q_1, $[e_1]$ and $[q_1]$ have the same image in $K_0(B)/kK_0(B)$ for $k = 1, 2, \ldots, K$. Therefore $(j_1)_* = (j_2 \circ \phi)_*$ on $K_0(F, \mathbf{Z}/k\mathbf{Z})$ for all $k \leqslant K$. Since $K_1(F) = K_1(C) = 0$, by the six-term exact sequence in 8.2 (see [32, 1.6]), both $[L]$ and $[\psi]$ map $K_0(A, \mathbf{Z}/k\mathbf{Z})$ to $K_0(B)/kK_0(B)$ and factor through $K_0(F, \mathbf{Z}/k\mathbf{Z})$. Therefore

$$[L]|_{G \cap K_0(A, \mathbf{Z}/k\mathbf{Z})} = [\psi]|_{G \cap K_0(A, \mathbf{Z}/k\mathbf{Z})}, \quad k = 1, 2, \ldots, K.$$

The general case in which F is a direct sum of $M_l(C([0,1]))$ follows immediately. □

9.9. Lemma. *Let $C = \bigoplus_{j=1}^n C_j$, where each $C_j = P_j M_{s(j)}(C(X_j))P_j$, P_j is a projection in $M_{s(j)}(C(X_j))$ and X_j is a path connected compact metric space with finitely generated $K_1(C_j)$, $K_0(C(X_j)) = \mathbf{Z} \oplus tor(K_0(C_j))$, $K_1(C(X_j))$ and $K_0(C_j)) \subset \{(z, x)\colon z \in \mathbf{N}$, or $(z, x) = (0, 0)\}$. Then C is KK-attainable.*

Proof. Clearly, by considering each summand separately, we may assume that C has only one summand. It is also clear that one can reduce the general case to the case in which $C = M_k(C(X))$ and $K_i(C)$ satisfies the condition described in the lemma.

Let A be a unital simple C^*-algebra with $TR(A) \leqslant 1$ and $\alpha \in KL(C, A)^{++}$. Let $\mathcal{P} \subset \mathbf{P}(A)$ be a finite subset and G be the subgroup generated by $\mathcal{P}$. By [11], for any finite subset $\mathcal{G} \subset C$ and $\eta > 0$, there exists a $\mathcal{G}$-$\eta/2$-multiplicative contractive completely positive linear map $\psi : C \to M_N(B)$ for some large integer N such that

$$[\psi]|_G = \alpha|_G + h|_G$$

for some one-point evaluation (at ζ) $h : C \to M_N(B)$. Since $TR(M_N(B)) \leqslant 1$, for any $\sigma_1 > 0$ and $\varepsilon/N > \sigma_2 > 0$, there exist a projection $p \in M_N(B)$ and a unital C^*-subalgebra $F \subset M_N(B)$ with $F \in \mathcal{I}$ and with $1_F = p$ such that:

(1) there are $\mathcal{G}$-η-multiplicative contractive completely positive linear maps $L_1 : C \to F$ and $L_2 : C \to (1-p)M_N(B)(1-p)$ such that

$$\left\| \psi(x) - L_1(x) \oplus L_2(x) \right\| < \sigma_1$$

for all $x \in \mathcal{G}$, and

(2) $\tau(1-p) < \sigma_2$ for all $\tau \in T(M_N(B))$.

With sufficiently small σ_1, we may assume that

$$[\psi]_G = [L_1]|_G + [L_2]|_G.$$

Suppose that

$$G \cap K_0(C, \mathbf{Z}/k\mathbf{Z}) = \{0\} \quad \text{for } k > K.$$

By Lemma 9.8, there exist a projection $e \leqslant p$ with $\tau(e) < \sigma_2$ for all $\tau \in T(M_N(B))$ and a unital $\mathcal{G}$-η-multiplicative contractive completely positive linear map $L_1' : C \to F_1$, where F_1 is a C^*-subalgebra of $pM_N(B)p$ with $1_{F_1} = e$ such that $\dim F_1 < \infty$ and

$$[L_1']|_{G\cap K_0(C,\mathbf{Z}/k\mathbf{Z})} = [L_1]|_{G\cap K_0(C,\mathbf{Z}/k\mathbf{Z})}$$

for all k so that $G \cap K_0(C, \mathbf{Z}/k\mathbf{Z}) \neq \{0\}$. So, in particular,

$$\big([L_1'] + [L_2]\big)|_{G\cap K_0(C,\mathbf{Z}/k\mathbf{Z})} = \alpha|_{G\cap K_0(C,\mathbf{Z}/k\mathbf{Z})} + h|_{G\cap K_0(C,\mathbf{Z}/k\mathbf{Z})}.$$

Without loss of generality, we may assume that $\alpha([1_C]) = [1_B]$. Note that, with a small σ_2, we have $[(1-p)+e] \leqslant [1_B]$. Let $q \in pM_N(B)$ be a projection with $[q] = [1_B] - [(1-p)+e]$. By applying an inner automorphism on $M_N(B)$ if necessary, without loss of generality, we may assume that $p, e, q \leqslant 1_B$. Let $h_0 : C \to qBq$ be the unital one-point evaluation (at ζ) (mapping 1_B to q).

We now define $\Psi = L_1' \oplus L_2 \oplus h_0$. Note if D is a C^*-algebra of finite dimension or $D \in \mathcal{I}$, then $K_1(D, \mathbf{Z}/k\mathbf{Z}) = 0$. Since the images of L_1' and h_0 in some C^*-algebras belong to $\mathcal{I}$,

$$\big([L_1'] + [h_0]\big)|_{G\cap K_1(C)} = 0, \qquad \big([L_1'] + [h_0]\big)|_{G\cap K_1(C,\mathbf{Z}/k\mathbf{Z})} = 0 \quad \text{and}$$
$$\big([L_1'] + [h_0]\big)|_{G\cap tor(K_0(C))} = 0.$$

Hence we compute that

$$[\Psi]|_{G\cap K_i(C)} = \alpha|_{G\cap K_i(C)}, \quad i = 0, 1, \quad \text{and} \quad [\Psi]|_{G\cap K_1(C,\mathbf{Z}/k\mathbf{Z})} = \alpha_{G\cap K_1(C,\mathbf{Z}/k\mathbf{Z})}.$$

The proof is complete if we can show, in addition, that $[\Psi]|_{G\cap K_0(C,\mathbf{Z}/k\mathbf{Z})} = \alpha|_{G\cap K_0(C,\mathbf{Z}/k\mathbf{Z})}$. We note that there is an (unnatural) splitting short exact sequence

$$0 \to K_0(C)/kK_0(C) \to K_0(C, \mathbf{Z}/k\mathbf{Z}) \to Tor\big(K_1(C, \mathbf{Z}/k\mathbf{Z})\big) \to 0.$$

From $[\Psi]|_{G\cap K_0(C)} = \alpha|_{G\cap K_0(C)}$, we conclude that

$$[\Psi]|_{G\cap K_0(C)/kK_0(C)} = \alpha|_{G\cap K_0(C)/kK_0(C)}.$$

On the other hand, it is easy to compute that

$$K_0(C, \mathbf{Z}/k\mathbf{Z}) = \mathbf{Z}/k\mathbf{Z} \oplus K_0\big(C_0(Y), \mathbf{Z}/k\mathbf{Z}\big),$$

where Y is the locally compact space formed by taking the point ζ away from X and the summand $\mathbf{Z}/k\mathbf{Z} \subset K_0(C)/kK_0(C)$. Since both h and h_0 are point-evaluation at ζ, we have

$$[h]|_{K_0(C_0(Y),\mathbf{Z}/k\mathbf{Z})} = [h_0]|_{K_0(C_0(Y),\mathbf{Z}/k\mathbf{Z})} = 0.$$

It follows that

$$[\Psi]_{G\cap K_0(C,\mathbf{Z}/k\mathbf{Z})}=\alpha|_{G\cap K_0(C,\mathbf{Z}/k\mathbf{Z})}.$$

This proves the lemma. □

Let X be a connected compact metric space and let $A=C(X)$. Denote by $\rho_A: K_0(A)\to\mathbf{Z}$ the rank map. In 9.9, we assume that $\ker\rho_{C(X_j)}$ is a finite group. One should note that the proof of 9.9 does not work when $\ker\rho_{C(X_j)}$ contains an infinite cyclic subgroup. This happens because L_1' would not kill $\ker\rho_{C(X_j)}$. In 9.10 below, we apply a result of L. Li and a result of Villadsen to avoid this problem.

Recall a C^*-algebra A is said to be locally AH if for any $\varepsilon>0$ and any finite subset $\mathcal{F}\subset A$, there exists a C^*-subalgebra $B\subset A$ with $B=PM_l(C(X))P$ for some compact metric space X and where P is a projection in $M_l(C(X))$, such that

$$\operatorname{dist}(x,B)<\varepsilon\quad\text{for all } x\in\mathcal{F}.$$

9.10. Proposition. *Let A be a separable unital simple C^*-algebra with $TR(A)\leqslant 1$. If A is locally AH, then A is pre-classifiable.*

Proof. It follows from [39] that A satisfies the AUCT. We may assume that $A=\overline{\bigcup_{n=1}^{\infty}A_n}$, where each A_n is a finite direct sums of $P_{n,i}M_{r(n,i)}C(X_{n,i})P_{n,i}$ and $X_{n,i}$ is a path connected finite CW complex. One may assume that $1_{A_n}=1_A$. Put $j_n: A_n\to A$ the embedding. Consider $j_n\times\alpha$. If A_n has only one summand, then $K_0(A_n)=\mathbb{Z}\oplus\ker\rho_{A_n}$. Since $\alpha\in KL(A,B)^{++}$, $(j_n\times\alpha)\in KL(A_n,B)^{++}$. By considering each summand separately, we may assume A_n has only one summand. Since A is simple, by 9.7 and 9.1, it suffices to show that, $A=C(X)$ is KK-attainable for every path connected finite CW complex X.

Let $\alpha\in KK(A,B)^{++}$. Suppose that $\alpha(1_A)=[p]$ $(\neq 0)$, where $p\in M_l(B)$ is a projection. Fix a unital nuclear simple C^*-algebra B with $TR(B)\leqslant 1$. By [57], there is a unital simple C^*-algebra C which is direct limit of C^*-algebras in 9.9 such that

$$\big(K_0(C),K_0(C)_+,[1_C],K_1(C)\big)=\big(K_0(B),K_0(B)_+,[1_A],K_1(B)\big).$$

By [52], there exists $\beta\in KK(C,B)$ which gives the above isomorphism.

Let $\alpha\in KL(A,B)^{++}$ and $\gamma=\alpha\times\beta^{-1}\in KL(A,C)^{+}$. Since $K_i(C(X))$ is finitely generated, $KL(A,C)=KK(A,C)$. In particular, $\gamma(K_0(A)_+\setminus\{0\})\subset K_0(C)_+\setminus\{0\}$. By [28], there is a homomorphism $h: A\to pM_l(C)p$ such that $[h]=\gamma$. But by 9.9, since each C^*-algebra described in 9.9 is KK-attainable, C is KK-attainable (see 9.1). Let $\varepsilon>0$ and fix finite subsets $\mathcal{F}\subset A$ and $\mathcal{P}\subset\mathbf{P}(A)$. Let $\mathcal{G}=h(\mathcal{F})\subset C$ and $\mathcal{Q}=[h](\mathcal{P})\subset\mathbf{P}(C)$. Let $\Lambda: C\to B$ be a $\mathcal{G}$-ε-multiplicative contractive completely positive linear map such that

$$[\Lambda]|_{\mathcal{Q}}=\beta|_{\mathcal{Q}}.$$

Define $L=\Lambda\circ h$. Then $L: A\to B$ is a $\mathcal{F}$-ε–multiplicative contractive completely positive linear map such that

$$[L]|_{\mathcal{P}}=\alpha|_{\mathcal{P}}.$$

So A is KK-attainable. □

9.11. Lemma. *Let A be a unital separable C^*-algebra, $\{\mathcal{F}_k\}$ be an increasing sequence of finite subsets of the unit ball of A such that $\bigcup_k \mathcal{F}_k$ is dense in the unit ball of A, and let $\Phi_n : A \to A$ be a sequence of unital contractive completely positive linear maps such that* $\lim_{n\to\infty} \|\Phi_n(a)\| = \|a\|$ *for all $a \in A$ and*

$$\sum_{k=n}^{\infty} \big\| \Phi_k(ab) - \Phi_k(a)\Phi_k(b) \big\| < \sum_{k=n}^{\infty} \delta_n,$$

for all $a, b \in \mathcal{G}_n$, and for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, $[\Phi_n]|_{\mathcal{P}} = [\mathrm{id}]|_{\mathcal{P}}$ for all sufficiently large n, where $\mathcal{G}_1 = \mathcal{F}_1$, $\mathcal{G}_{n+1} \supset \bigcup_{k=1}^{n} \Phi_k(\mathcal{F}_n) \cup \mathcal{F}_n \cup \Phi_n(\mathcal{G}_n)$, $n = 1, 2, \ldots,$ and where $\sum_{n=1}^{\infty} \delta_n < \infty$. Let $B = \lim_{n\to\infty}(A, \Phi_n)$ be the generalized inductive limit in the sense of [4]). *Then $\{\Phi_n\}$ induces an isomorphism*

$$\big(K_0(B), K_0(B)_+, [1_B], K_1(B)\big) = \big(K_0(A), K_0(A)_+, [1_A], K_1(A)\big).$$

Proof. The proof is standard. We sketch here. Write $K_i(A) = \bigcup_{n=1}^{\infty} G_n^{(i)}$, where each $G_n^{(i)}$ is a finitely generated subgroup of $K_0(A)$. Let $\Phi_{n,n+m} = \Phi_{n+m} \circ \Phi_{n+m-1} \circ \cdots \circ \Phi_n$ and $\Psi_n : A \to B$ be the map induced by the inductive system which maps the nth A to B. For each $G_n^{(i)}$, we may assume that $[\Psi_m]|_{G_n^{(i)}}$ is well defined for all $m \geqslant n$. The assumption that $[\Phi_n]|_{\mathcal{P}} = [\mathrm{id}_A]|_{\mathcal{P}}$ for all sufficiently large n implies that $[\Psi_m]|_{G_n^{(i)}} = [\Psi_{m'}]|_{G_n^{(i)}}$ for all $m, m' \geqslant n$. This gives a homomorphism $\beta_i : K_i(A) \to K_i(B)$ $(i = 0, 1)$.

Suppose that $p_1, p_2, v \in M_l(B)$ such that $v^*v = p_1$ and $vv^* = p_2$. There is a sequence $\{\Psi_{n_k}(a_k)\}$, where $a_k \in M_l(A)$, such that it converges to v. Since $v^*v = p_1$, we have $\Psi_{n_k}(a_k^* a_k) \to p_1$ and $\Psi_{n_k}(a_k a_k^*) \to p_2$. Therefore we may assume that

$$\big\|\Psi_{n_k}\big(a_k^* a_k - \big(a_k^* a_k\big)^2\big)\big\| < 1/2^{k+1} \quad \text{and} \quad \big\|\Psi_{n_k}\big(a_k^* a_k\big) - p_1\big\| < 1/2^{k+1}.$$

Since $\|\Psi_m(x)\| = \limsup \|\Phi_{m,n}(x)\|$ for all $x \in A$ and $m \geqslant 1$, by passing to a subsequence and possibly replacing a_k by $\Phi_{n_k,m_k}(a_k)$, Ψ_{n_k} by Ψ_{m_k}, if necessary, we may assume that

$$\big\|a_k^* a_k - \big(a_k^* a_k\big)^2\big\| < 1/2^k, \quad k = 1, 2, \ldots.$$

It is standard that there is a partial isometry v_k and a projection $q_k \in A$ such that

$$v_k^* v_k = q_k \quad \text{and} \quad \|v_k - a_k\| < 1/2^{k-1}$$

for all large k. Let $q_k' = v_k v_k^*$. Note also, for any $\varepsilon > 0$, we have

$$\big\|\Psi_{n_k}(q_k) - p_1\big\| < \varepsilon \quad \text{and} \quad \big\|\Psi_{n_k}\big(q_k'\big) - p_2\big\| < \varepsilon$$

for all large k. Hence $[\Psi_{n_k}](q_k) = [p_1]$ and $[\Psi_{n_k}](q_k') = [p_2]$. This, in particular, implies that $[p_1]$ is in the image of β_0. It follows that β_0 is surjective. Note also that $[q_k] = [q_k']$ in $K_0(A)$. It follows that β_0 is also injective. It is also easy to check from the definition that β_0 preserves the order.

In the above, if we let $w^*w = p_1$ and $ww^* \leqslant p_2$, then exactly the same argument shows that there are partial isometries $v_k \in A$ such that $v_k v_k^* = q_k$, $v_k v_k^* \leqslant q_k'$ and $\Psi_{n_k}(v_k) \to v$, $\Psi_{n_k}(q_k) \to p_1$ and $\Psi_{n_k}(q_k') \to vv^* \leqslant p_2$. These imply that β_0 is an order isomorphism.

A similar argument shows that β_1 is an isomorphism and $K_1(A) = K_1(B)$. □

9.12. Theorem. *Let A be a unital separable nuclear simple C^*-algebra with $TR(A) \leqslant 1$ satisfying the* AUCT. *Then there exists a unital separable nuclear simple C^*-algebra B with $TR(B) = 0$ satisfying* AUCT *and the following*:

(1) $(K_0(A), K_0(A)_+, [1_A], K_1(A)) = (K_0(B), K_0(B)_+, [1_B], K_1(B))$,
(2) *there exists a sequence of contractive completely positive linear maps $\Phi_n : A \to B$ such that*:
 (i) $\lim_{n\to\infty} \|\Phi_n(ab) - \Phi_n(a)\Phi_n(b)\| = 0$ *for $a, b \in A$*,
 (ii) *For each finite subset $\mathcal{P} \subset \mathbf{P}(A)$ there exists an integer $N > 0$ such that*

$$[\Phi_n]|_{\mathcal{P}} = [\alpha]|_{\mathcal{P}}$$

for all $n \geqslant N$, where $\alpha \in KL(A, B)$ which gives an identification in (1) *above.*

Proof. Let $\{\mathcal{P}_n\}$ be an increasing sequence of finite subsets of $\mathbf{P}(A)$ such that the union is dense in $\mathbf{P}(A)$. In particular the union of the subgroups generated by the images of $\mathcal{P}_n$ in $\underline{K}(A)$ is $\underline{K}(A)$. Let $\{\mathcal{F}_n\}$ be an increasing sequence of finite subsets of the unit ball of A whose union is dense in the unit ball of A. Let $\{\delta_n\}$ be a decreasing sequence of positive numbers such that $\sum_{n=1}^{\infty} \delta_n < \infty$. Without loss of generality, we may assume that any $\mathcal{F}_n$-δ_n-multiplicative contractive completely positive linear map L_n defined on A well defines $[L_n]|_{\mathcal{P}_n}$. Furthermore we may assume that $(\mathcal{F}_n, \varepsilon_n, \mathcal{G}_n, \mathcal{P}_n, \delta_n)$ (for some finite subsets $\mathcal{G}_n$ and $\mathcal{P}_n$) forms a 5-tuple as defined in 6.8 of [39] for $l = 1$, $b \geqslant \pi$ and $M = 1$. We may also assume that $\mathcal{F}_n \subset \mathcal{G}_n$.

Let $\mathcal{G}_1'$ be the union of $\mathcal{G}_1$ and $\{(a + a^*)_+, (a + a^*)_-, (a - a^*)_+, (a - a^*)_-\colon a \in \mathcal{F}_1\}$. Since A is simple, if $b \in (\mathcal{G}_1')_+$ is a nonzero element, there are $x_i(b) \in A$ such that

$$\sum_{i=1}^{n(b)} x_i(b)^* b x_i(b) = 1_A.$$

Let $\mathcal{G}_1''$ be the union of $\mathcal{G}_1'$ and $\{x_i(b), x_i(b)^*\colon b \in (\mathcal{G}_1')_+\}$. Since $TR(A) \leqslant 1$, there exists a C^*-subalgebra $C_1 \in \mathcal{I}$ with $1_{C_1} = p_1$ such that:

(i$_1$) $\|p_1 a - a p_1\| < \delta_1/4$ for all $a \in \mathcal{G}_1''$,
(ii$_1$) $p_1 a p_1 \in_{\delta_1/4} C_1$ and $\|(1 - p_1)a(1 - p_1)\| \geqslant (1 - \delta_1/4)\|a\|$ for all $a \in \mathcal{G}_1''$ (see 5.6) and
(iii$_1$) $\tau(1 - p_1) < \delta_1/4$ for all $\tau \in T(A)$.

We may also assume that:

(iv$_1$) $\|\sum_{i=1}^{n(b)} (1 - p_1)x_i(b)^*(1 - p_1)b(1 - p_1)x_1(b)(1 - p_1) - (1 - p_1)\| < 1/8$ and there are $z_i(b), b' \in C_1$ such that $\|b - b'\| < \delta_1/2$ and $\|\sum_{i=1}^{n(b)} z_i(b)^* b' z_i(b)^* - p_1\| < 1/8$ for all $b \in (\mathcal{G}_1')_+$.

Write $C_1 = M_{d(1,1)}(C([0,1])) \oplus \cdots \oplus M_{d(s_1,1)}(C([0,1]) \oplus M_{d(s_1+1,1)} \oplus \cdots \oplus M_{d(t_1,1)}$. Put $F_1 = M_{d(1,1)} \oplus \ldots \oplus M_{d(t_1,1)}$. Note that $\dim F_1 < \infty$. Define $\pi_1 : C_1 \to F_1$ to be a surjective point-evaluation map. We identify F_1 with the unital C^*-subalgebra of C_1 (scalar matrices). So $F_1 \subset C_1 \subset p_1 A p_1$. Note $\underline{K}(C_1) = \underline{K}(F_1)$ and $[\pi_1]$ gives the identification. Since A is nuclear, there is a contractive completely positive linear map $L_1' : p_1 A p_1 \to C_1$ such that

$$\|L_1'(p_1 a p_1) - p_1 a p_1\| < \delta_1/4 \quad \text{for all } a \in \mathcal{G}_1''.$$

Define $L_1(a) = \pi_1 \circ L_1'(p_1 a p_1)$ for $a \in A$. Then L_1 is $\mathcal{G}_1''$-$\delta_1/2$-multiplicative. Define $\phi_1 : A \to A$ by $\phi_1(a) = (1-p_1)a(1-p_1) \oplus L_1(a)$ for $a \in A$. It is clear that ϕ_1 is $\mathcal{F}_1$-$\delta_1/2$-multiplicative. Furthermore,

$$[\phi_1]|_{\mathcal{P}_1} = [\mathrm{id}_A]|_{\mathcal{P}_1}, \qquad \|\phi_1(a)\| \geqslant (1-\delta_1/2)\|a\| \quad \text{for } a \in \mathcal{F}_1 \quad \text{and}$$
$$\left\|\sum_{i=1}^{n(b)} \phi_1\big(x_i(b)\big)^* \phi_1(b) \phi_1\big(x_i(b)\big) - 1_A\right\| < 1/4.$$

Let $\mathcal{G}_2'$ be the union of $\mathcal{G}_2$, $\phi_1(\mathcal{G}_1'')$ and $\{(a+a^*)_+, (a+a^*)_-, (a-a^*)_+, (a-a^*)_-\colon a \in \mathcal{F}_2 \cup \phi_1(\mathcal{G}_1'')\}$. Since A is simple, for each nonzero positive element $b \in \mathcal{G}_2'$, there are $x_1(b), \ldots, x_{n(b)}(b) \in A$ such that

$$\sum_{i=1}^{n(b)} x_i(b)^* b x_i(b) = 1_A.$$

Now let $\mathcal{G}_2''$ be a finite subset of A containing $\mathcal{G}_2'$, $\{1-p_1, p_1\}$, a generating set of F_1 and $\{x_i(b), x_i(b)^*\colon b \in (\mathcal{G}_2')_+\}$. Let $\mathcal{Q}_2$ be the finite subset which is the union of $\mathcal{P}_2$, $1-p_1$, p_1 and contains at least one minimal projection of each summand of F_1, By choosing a possibly larger $\mathcal{G}_2''$ and smaller δ_2 we may assume that any $\mathcal{G}_2''$-δ_2-multiplicative contractive completely positive linear map L well defines $[L]|_{\mathcal{Q}_2}$. Since $TR(A) \leqslant 1$, there is a C^*-subalgebra $C_2 \in \mathcal{I}$ with $1_{C_2} = p_2$ such that:

(i$_2$) $\|p_2 a - a p_2\| < \delta_2/4$ for all $a \in \mathcal{G}_2''$,
(ii$_2$) $p_2 a p_2 \in_{\delta_2/4} C_2$ and $\|(1-p_2)a(1-p_2)\| \geqslant (1-\delta_2/4)\|a\|$ for all $a \in \mathcal{G}_2''$ (see 5.6),
(iii$_2$) $\tau(1-p_2) < \delta_2/4$ for all $\tau \in T(A)$, and
(iv$_2$) $\|\sum_{i=1}^{n(b)}(1-p_2)x_i(b)^*(1-p_2)b(1-p_2)x_i(b)(1-p_2) - (1-p_2)\| < 1/32$ and there are $z_i(b), b' \in C_2$ such that $\|b-b'\| < \delta_1/2$ and $\|\sum_{i=1}^{n(b)} z_i(b)^* b' z_i(b)^* - p_2\| < 1/32$ for all $b \in (\mathcal{G}_2')_+$.

Write $C_2 = M_{d(1,2)}(C([0,1])) \oplus \cdots \oplus M_{d(s_2,2)}(C([0,1])) \oplus M_{d(s_2+1,2)} \oplus \cdots \oplus M_{d(t_2,2)}$. Put $F_2 = M_{d(1,2)} \oplus \ldots \oplus M_{d(t_2,2)}$. Note that $\dim F_2 < \infty$. Define $\pi_2 : C_2 \to F_2$ to be a surjective point-evaluation map. We identify F_2 with the unital C^*-subalgebra of C_2 (scalar matrices). So $F_2 \subset C_2 \subset p_2 A p_2$. Note that $\underline{K}(C_2) = \underline{K}(F_2)$ and $[\pi_2]$ gives the identification. Since A is nuclear, there is a contractive completely positive linear map $L_2' : p_2 A p_2 \to C_2$ such that

$$\|L_2'(p_2 a p_2) - p_2 a p_2\| < \delta_1/4 \quad \text{for all } a \in \mathcal{G}_2''.$$

Define $L_2(a) = \pi_2 \circ L'_2(p_2ap_2)$ for $a \in A$. Then L_2 is $\mathcal{F}_2$-$\delta_2/2$-multiplicative. Define $\phi_2 : A \to A$ by $\phi_2(a) = (1-p_2)a(1-p_2) \oplus L_2(a)$ for $a \in A$. It is clear that ϕ_2 is $\mathcal{F}_2$-$\delta_2/2$-multiplicative. Furthermore,

$$[\phi_2]|_{\mathcal{Q}_2} = [\mathrm{id}_A]|_{\mathcal{Q}_2}, \qquad \left\|\phi_2(a)\right\| \geqslant (1-\delta_2/4)\|a\| \quad \left(a \in \mathcal{G}''_2\right) \quad \text{and}$$
$$\left\|\sum_{j=1}^{n}(b)\phi_2\left(x_j(b)\right)\phi_2(b)\phi_2\left(x_j(b)\right) - 1_A\right\| < 1/16.$$

Since $\dim F_1 < \infty$, we may also assume that there exists an injective homomorphism $h_2 : F_1 \to A$ such that

$$\left\|h_2 - (L_2)|_{F_1}\right\| < 1/4$$

(see for example [37, 2.3]). We continue the construction of L_n, π_n and ϕ_n in this fashion.

Let $\mathcal{G}'_{n+1}$ be the union of $\mathcal{G}_{n+1}$, $\phi_1(\mathcal{G}''_n), \dots, \phi_n(\mathcal{G}''_n)$ and $\{(a+a^*)_+, (a+a^*)_-, (a-a^*)_+, (a-a^*)_-\colon a \in \mathcal{F}_n \bigcup_{k=1}^n \phi_k(\mathcal{G}''_n)\}$. Since A is simple, for each nonzero positive element $a \in \mathcal{G}'_{n+1}$, there are $x_1(a), \dots, x_{n(a)}(a) \in A$ such that $\sum_{i=1}^{n(a)} x_i(a)^* a x_i(a) = 1_A$.

Now let $\mathcal{G}''_{n+1}$ be a finite subset of A containing $\mathcal{G}'_{n+1}$, $\{1-p_n, p_n\}$, a generating set of F_n and $\{x_i(b), x_i(b)^*\colon b \in (\mathcal{G}'_{n+1})_+\}$. Let $\mathcal{Q}_{n+1}$ be the finite subset which is the union of $\mathcal{P}_{n+1}$, $1-p_{n+1}$, p_{n+1} and at least one minimal projection of each summand of F_n. By choosing a possibly larger $\mathcal{G}''_{n+1}$ and smaller δ_n we may assume that any $\mathcal{G}''_{n+1}$-δ_n-multiplicative contractive completely positive linear map L well defines $[L]|_{\mathcal{Q}_n}$.

Since $TR(A) \leqslant 1$, there is a C^*-subalgebra $C_{n+1} \in \mathcal{I}$ with $1_{C_{n+1}} = p_{n+1}$ such that:

(i_{n+1}) $\|p_{n+1}a - ap_{n+1}\| < \delta_{n+1}/4$ for all $a \in \mathcal{G}''_n$,
(ii_{n+1}) $p_{n+1}ap_{n+1} \in_{\delta_{n+1}/4} C_{n+1}$ and $\|(1-p_{n+1})a(1-p_{n+1})\| \geqslant (1-\delta_{n+1}/4)\|a\|$ for all $a \in \mathcal{G}''_n$,
(iii_{n+1}) $\tau(1-p_{n+1}) < \delta_n/4$ for all $\tau \in T(A)$ and
(iv_{n+1}) $\|\sum_{i=1}^{n(b)}(1-p_{n+1})x_i(b)^*(1-p_{n+1})b(1-p_{n+1})x_i(b)(1-p_{n+1}) - (1-p_{n+1})\| < 1/2^{n+3}$ and there are $z_i(b), b' \in C_2$ such that

$$\|b-b'\| < \delta_{n+1}/2 \quad \text{and} \quad \left\|\sum_{i=1}^{n(b)} z_i(b)^* b' z_i(b)^* - p_{n+1}\right\| < 1/2^{n+3} \quad \text{for all } b \in (\mathcal{G}'_{n+1})_+.$$

Write $C_{n+1} = M_{d(1,n+1)}(C([0,1])) \oplus \cdots \oplus M_{d(s_{n+1},n+1)}(C([0,1]) \oplus M_{d(s_{n+1}+1,n+1)} \oplus \cdots \oplus M_{d(t_{n+1},n+1)}$. Put $F_{n+1} = M_{d(1,n+1)} \oplus \ldots \oplus M_{d(t_{n+1},n+1)}$. Note that $\dim F_{n+1} < \infty$. Define $\pi_{n+1} : C_{n+1} \to F_{n+1}$ to be a surjective point-evaluation map. We identify F_{n+1} with the unital C^*-subalgebra of C_{n+1} (scalar matrices). So $F_{n+1} \subset C_{n+1} \subset p_{n+1}Ap_{n+1}$. Note that $\underline{K}(C_{n+1}) = \underline{K}(F_{n+1})$ and $[\pi_{n+1}]$ gives the identification. Since A is nuclear, there is a contractive completely positive linear map $L'_{n+1} : p_{n+1}Ap_{n+1} \to C_{n+1}$ such that

$$\left\|L'_{n+1}(p_{n+1}ap_{n+1}) - p_{n+1}ap_{n+1}\right\| < \delta_n/4 \quad \text{for all } a \in \mathcal{F}_{n+1}.$$

Define $L_{n+1}(a) = \pi_{n+1} \circ L'_{n+1}(p_{n+1}ap_{n+1})$, $a \in A$. Then L_{n+1} is $\mathcal{F}_{n+1}$-$\delta_{n+1}/2$-multiplicative. Define $\phi_{n+1} : A \to A$ by $\phi_{n+1}(a) = (1 - p_{n+1})a(1 - p_{n+1}) \oplus L_{n+1}(a)$. Then ϕ_{n+1} is $\mathcal{G}''_{n+1}$-δ_n-multiplicative. Furthermore,

$$[\phi_{n+1}]|_{\mathcal{Q}_{n+1}} = [\mathrm{id}_A]|_{\mathcal{Q}_{n+1}}, \qquad \|\phi_{n+1}(a)\| \geqslant (1 - \delta_{n+1}/2)\|a\| \quad \text{for all } a \in \mathcal{G}''_{n+1} \quad \text{and}$$
$$\left\| \sum_{j=1}^{n(b)} \phi_{n+1}\big(x_j(b)\big)\phi_{n+1}(b)\phi_{n+1}\big(x_j(b)\big) - 1_A \right\| < 1/2^{n+1}.$$

Again, we may also assume that there exists an injective homomorphism $h_{n+1} : F_n \to A$ such that

$$\|h_{n+1} - (L_{n+1})|_{F_n}\| < 1/2^{n+1}.$$

We then define $B = \lim_n(A, \phi_n)$. This is a generalized inductive limit in the sense of [4] (but $\{\phi_n\}$ is in fact asymptotically multiplicative). Note that B is a unital separable C^*-algebra. By [4, 5.13], B is nuclear. From (iv_{n+1}) and the construction above, it is easy to check that B is simple. Since for each n

$$[\phi_{n+1}]|_{\mathcal{Q}_{n+1}} = [\mathrm{id}_A]|_{\mathcal{Q}_{n+1}},$$

by 9.11, we have that

$$\big(K_0(B), K_0(B)_+, [1_B], K_1(B)\big) = \big(K_1(A), K_1(A)_+, [1_A], K_1(A)\big).$$

From the construction it is also standard to show that $TR(B) = 0$ (see for example the proof of [31, 4.3] and also [37]). Let $\phi_{n,\infty} : A \to B$ be the map from the nth A to B induced by the inductive limit system. Set also $\Phi_n = \phi_{n,\infty}$. It is clear $\{\Phi_n\}$ satisfies (i) and (ii). To see that B satisfies AUCT, we note that B satisfies the property (P) described in [39, 6.8]. It follows from [39, 6.13 and 6.16] that B also satisfies the AUCT. □

10. The classification theorem

To establish the classification of separable nuclear simple C^*-algebras with tracial rank no more than one, we first present, for each Elliott invariant satisfying the abstract conditions described in 10.1 below, a concrete AH-algebra (with special form) with no dimension growth realizing the given Elliott invariant (see Theorem 10.1). We then show that a separable simple nuclear C^*-algebra A with $TR(A) \leqslant 1$ has an Elliott invariant which coincides with the Elliott invariant of one of the aforementioned concrete AH-algebras, and prove that if A satisfies (AUCT), then it is in fact isomorphic to the said concrete AH-algebra.

The following theorem was proved by J. Villadsen. The extra conditions (1) and (2) are not new either. It has appeared implicitly in several places including Villadsen's proof.

If X is a convex set, the extremal points of X are denoted by $\partial_e(X)$.

10.1. Theorem. *(Cf. [57].) Suppose that G is a countable, partially ordered abelian group which is simple, weakly unperforated with the Riesz interpolation property, that $G/tor(G)$ is non-cyclic,*

$u \in G_+$, H is a countable abelian group, Δ is a metrizable Choquet simplex and $\lambda : \Delta \to S(G, u)$ is a continuous affine map with $\lambda(\partial_e \Delta) = \partial_e S(G, u)$. Then there is a simple AH-algebra $A = \lim_{n\to\infty}(A_n, h_n)$ with $TR(A) \leqslant 1$ and with $A_n = C_1 \oplus C_2 \oplus \cdots \oplus C_{m(n)}$, where C_1 is of the form as described in 7.1 (*a single summand*) *and C_j is of the form $C([0, 1]) \otimes M_{m(j)}$* (*for $j > 1$*)*, such that*:

(1) *$h_n = h_n^{(0)} \oplus h_n^{(1)} \oplus h_n^{(2)}$, where $h_n^{(0)}$, $h_n^{(1)}$ factor through a C^*-algebra in $\mathcal{I}$, and h_n is injective, in particular, $h_n^{(0)}$ is homotopically trivial,*
(2) *$\tau \circ h_{n+1,\infty} \circ h_n^{(0)}(1_{A_n}) \to 0$ uniformly on $T(A)$,*
(3) *$\tau \circ h_{n+1,\infty} \circ h_n^{(2)}(1_{A_n}) \to 0$ uniformly on $T(A)$,*
(4) *$(h_n)_{*1}$ is injective and*
(5) *$(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), r_A) = (G, G_+, u, H, \Delta, \lambda)$.*

Proof. The proof of this is a combination of Villadsen's proof of the main theorem in [57] and the proof of [36, 1.5]. Let $A = \lim_{n\to\infty}(A_n, h_n)$ be as in [36, 1.5]. This algebra A satisfies (3), (4) and $(K_0(A), K_0(A)_+, [1_A], K_1(A)) = (G, G_+, u, H)$. Moreover each A_n can be chosen so it has the form as required. Here one needs one modification and one explanation. In the proof of [36, 1.5] we use $C_j = M_{m(j)}$ for $j > 1$. But we can map $C([0, 1], M_{m(j)})$ into $M_{m(j)}$ by a one point-evaluation and then map $M_{m(j)}$ into $C([0, 1], M_{m(j)})$ (as constant functions). So we can assume that A_n has the required form. Note also that the new A_n has the same K-theory as the old one. If $K_1(A) = F = \lim_{n\to\infty}(F_n, \gamma_n)$, in the proof of [36, 1.5], $K_1(A_n) = F_n$ and the map $\Phi_{n,n+1}$ in the proof of [36, 1.5] has the property $(\Phi_{n,n+1})_{*1} = \gamma_n$. However, since F is a countable abelian group, one can always assume that F_n is finitely generated and γ_n is injective (by choosing F_n as subgroups and γ_n as embeddings) so that (4) holds.

We will revise the map h_n to meet the other requirements. Villadsen's proof in [57] is to replace h_n by ϕ_n without changing its K-theory in such a way that one gets Δ as tracial space and λ as pairing. We will follow his proof with a minor modification. Each h_n may be written as $h_n' \oplus h_n''$, where h_n'' is a point-evaluation, as in [36, 1.5]. Note that [57, Lemma] holds when X_q^i is a compact connected CW complex with dimension at least one but no more than three. Following Villadsen's proof, by applying [57, Lemma] and its proof, one can replace h_n'' to achieve exactly what [57, Lemma] achieved. It should be noted that Villadsen's proof of the main theorem in [57] works when X_q^j has lower dimension (but at least one), since the required maps $i_q^j : [0, 1] \to X_q^j$ and $k_q^j : X_q^j \to [0, 1]$ still exist. The new map obtained from Villadsen's proof has the form $\tilde{\psi}_n = h_n' \oplus \tilde{h}_n''$, where $\tilde{h}_n''$ is homotopically trivial. Furthermore, it can be chosen so that it factors through a C^*-algebra in $\mathcal{I}$. The construction of Villadsen then gives a simple AH-algebra B with $TR(B) \leqslant 1$ and satisfies (5). Moreover, one has $\tau \circ \tilde{\psi}_{n+1} \circ h_n(1_{A_n}) \to 0$ uniformly on $T(B)$. The construction does not change (3) and (4). It is also easy to get (1) and (2). For example, consider $h_{n+1}' \circ h_n' \oplus h_{n+1}' \circ \tilde{h}_n'' \oplus \tilde{h}_{n+1}'' \circ \tilde{\psi}_n$. Note that $h_{n+1}' \circ \tilde{h}_n''$ and $\tilde{h}_{n+1}'' \circ \tilde{\psi}_n$ are homotopically trivial and $\tau \circ \tilde{\psi}_{n,\infty} \circ h_{n+1}' \circ \tilde{h}_n''(1_{A_n}) \to 0$ uniformly on $T(B)$. □

10.2. Definition. Let C be a unital C^*-algebra. We denote by $S_u(K_0(C))$ the set of states on $K_0(C)$, i.e., the set of order and unit preserving homomorphisms from $K_0(C)$ to (the additive group) $\mathbb{R}$. There is an affine map $\lambda : T(C) \to S_u(K_0(C))$ such that $\lambda(t)([p]) = t(p)$ for all projections $p \in M_\infty(C)$ and $t \in T(C)$. Suppose that C is stably finite. It was proved in [51, Theorem 6.1] (for the simple case) and [6, Theorem 3.5] that each state in $S_u(K_0(C))$ is induced by a quasitrace $t \in QT(C)$. If C is exact, or if it is both simple and of tracial rank at most one, then all quasitraces on C are traces (see (ix) in 4.9).

Let A and B be two unital C^*-algebras. We say

$$\gamma : \big(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A)\big) \to \big(K_0(B), K_0(B)_+, [1_B], K_1(B), T(B)\big)$$

is an order isomorphism if there is an order isomorphism

$$\gamma_0 : \big(K_0(A), K_0(A)_+\big) \to \big(K_0(B), K_0(B)_+\big)$$

which maps $[1_A]$ to $[1_B]$, there is an isomorphism $\gamma_1 : K_1(A) \to K_1(B)$ and an affine homeomorphism $\gamma_2 : T(A) \to T(B)$ such that $\gamma_2^{-1}(\tau)(x) = \tau(\gamma_0(x))$ for all $\tau \in T(B)$ and $x \in K_0(A)$, where we view τ as a state on $K_0(A)$.

10.3. Theorem. *Let A and B be two unital separable nuclear simple C^*-algebras with $TR(A) \leqslant 1$ and $TR(B) \leqslant 1$ satisfying* AUCT *such that*

$$\big(K_0(B), K_0(B)_+, [1_B], K_1(B), T(B)\big) = \big(K_0(A), K_0(A), [1_A], K_1(A), T(A)\big)$$

in the sense of 10.2. *Then there is a sequence of contractive completely positive linear maps $\{\Psi_n\}$ from A to B such that*:

(i) $\lim_{n\to\infty} \|\Psi_n(ab) - \Psi_n(a)\Psi_n(b)\| = 0$ *for all $a, b \in A$,*

(ii) *for any finite subset set $\mathcal{P} \subset \mathbf{P}(A)$,*

$$[\Psi_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}},$$

for all sufficiently large n, where $\alpha \in KL(A, B)^{++}$ (see 8.2) *gives the above identification on K-theory and*

(iii) $$\lim_{n\to\infty} \sup_{\tau\in T(B)} \big\{\big|\tau \circ \Psi_n(a) - \xi\big(Q(a)\big)(\tau)\big|\big\} = 0$$

for all $a \in A_{\mathrm{sa}}$, where $\xi : \mathit{Aff}\, T(A) \to \mathit{Aff}\, T(B)$ is the affine isometry given above.

Proof. It follows from Theorem 9.12 that there is a unital separable simple nuclear C^*-algebra C with $TR(C) = 0$ satisfying the AUCT such that

$$\big(K_0(A), K_0(A), [1_A], K_1(A)\big) = \big(K_0(C), K_0(C)_+, [1_C], K_1(C)\big)$$

and a sequence of contractive completely positive linear maps $L_n : A \to C$ satisfying condition (2) in 9.12. In particular, $[L_n]|_{\mathcal{P}} = \beta|_{\mathcal{P}}$, for any finite subset $\mathcal{P}$ and all sufficiently large n, where $\beta \in KL(A, C)^{++}$ gives the above identification on K-theory. It follows from [38] that there is a unital separable simple AH-algebra C_1 such that $C_1 \cong C$. To simplify notation, we may assume that $C_1 = C$.

It follows from 9.10 that there exists a sequence of contractive completely positive linear maps $\Phi'_n : C \to B$ such that:

(i′) $\lim_{n\to\infty} \|\Phi'_n(ab) - \Phi'_n(a)\Phi'_n(b)\| = 0$ for all $a, b \in C$, (ii′) for any finite subset $\mathcal{Q} \subset \mathbf{P}(C)$,

$$[\Phi_n']|_{\mathcal{Q}} = (\beta^{-1} \times \alpha)|_{\mathcal{Q}}, \quad \text{for all sufficiently large } n.$$

Thus by choosing a subsequence $\{k(n)\}$ and defining $\Psi_n = \Phi'_{k(n)} \circ L_n : A \to B$ we see that Ψ_n satisfies (i) and (ii). (In fact one can show that A is KK-attainable.) We then apply the proof of 9.7, to obtain a (new) sequence $\{\Phi_n\}$ which also satisfies (iii). □

Using the argument of [38], Zhuang Niu gives a different proof of the above theorem.

10.4. Theorem. *Let A and B be two unital separable nuclear simple C^*-algebras with $TR(A) \leqslant 1$ and $TR(B) \leqslant 1$ satisfying the* AUCT. *Suppose that $\lambda(\partial_e(T(A))) = \partial_e(S_u(K_0(A)))$ and $\lambda(\partial_e(T(B))) = \partial_e(S_u(K_0(B)))$. Then A is isomorphic to B if and only if there exists an order isomorphism*

$$\gamma = (\gamma_0, \gamma_1, \gamma_2) : (K_0(A), K_0(A)_+, [1_A], K_1(A), T(A)) \to \\ (K_0(B), K_0(B)_+, [1_B], K_1(B), T(B)),$$

where $\gamma_2^{-1}(\tau)(x) = \tau(\gamma_0(x))$ for all $\tau \in T(B)$ and $x \in K_0(A)$ (see 10.2*).*

Proof. Let A be as in the theorem. Note that $T(A)$ is a Choquet simplex (see 4.9(ix) for example). By 10.1 and 10.3 (as well as 4.8), there is a unital simple AH-algebra $B = \lim_{n\to\infty}(B_n, \phi_{n,n+1})$ with

$$(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B), T(B)),$$

where B_n is as described in 10.1. Here $\phi_{n,n+1}$ is a homomorphism from B_n to B_{n+1}. Put $\phi_{n,m} = \phi_{m-1,m} \circ \cdots \circ \phi_{n,n+1}$. Denote by $\psi_n : B_n \to B$ the homomorphism induced by the inductive system. As in 10.1, we will assume that $\phi_{n,m}$ and ψ_n are injective and $(\phi_{n,m})_{*1}$ is injective for all $m > n$. In what follows, when it is convenient, we may identify B_n with $\psi_n(B_n)$ without further warning. We also assume that the inductive limit satisfies the conditions (1)–(5) in 10.1. To prove the theorem it suffices to prove that A is isomorphic to this specially constructed B. In what follows, κ is the quotient map from $U(D)/CU(D)$ to $K_1(D)$ for a C^*-algebra D.

Let $\xi : \mathit{Aff}\, T(A) \to \mathit{Aff}\, T(B)$ be the affine isometry induced by the above identification. Since both A and B satisfy the AUCT, there is $\alpha \in KL(A, B)^{++}$ which gives the isomorphism $(K_0(A), K_0(A)_+, [1_A], K_1(A)) \to (K_0(B), K_0(B)_+, [1_B], K_1(B))$. Let $\mathbf{L} : U(B) \to \mathbf{R}_+$ be defined as follows: if $u \in U_0(B)$, $\mathbf{L}(u) = 2\,\mathrm{cel}(u) + 8\pi + \pi/16$; if $u \in U(B) \setminus U_0(B)$ and $u^k \in U_0(B)$ (where k is smallest such positive integer) $\mathbf{L}(u) = 16\pi + 2\,\mathrm{cel}(u^k)/k + \pi/16$, and if $u \in U(B) \setminus U_0(B)$ and $[u]$ is not of finite order in $K_1(B)$, $\mathbf{L}(u) = 16\pi + \pi/16$. Fix any $\varepsilon > 0$ ($\varepsilon < \pi/128$) and finite subset $\mathcal{F} \subset B$. Let $\delta' > 0$, integer $n > 0$, finite subsets $\mathcal{P} \subset \mathbf{P}(B)$, $\mathcal{S} \subset B$ be as required in Theorem 8.6 (corresponding to B, $\mathbf{L}$, $\varepsilon/2$ and $\mathcal{F}$). There are mutually orthogonal projections $q, p_1, p_2, \ldots, p_n$ with $q \lesssim p_i$, and with $[p_i] = [p_1]$ ($i = 1, 2, \ldots, n$) and there is a C^*-subalgebra $C_1' \in \mathcal{I}$ with $1_{C_1'} = q$ and there are unital $\mathcal{S}$-$\delta'/4$-multiplicative contractive completely positive linear maps $h_0 : B \to qBq$ and $h_1 : B \to C_1'$ such that $h_0(x) = qxq$ and

$$\|x - (h_0(x) \oplus h_1(x) \oplus \cdots \oplus h_1(x))\| < \delta'/16$$

for all $x \in S$, where h_1 is repeated n times. Put $C = M_n(C_1') \subset (1-q)B(1-q)$. Let $\mathcal{P}_0$, $\mathcal{G}_0$, $\mathcal{H}$, $\delta_0 > 0$ and $\sigma_1 > 0$ also be as required by 8.6 and let $\delta = \min\{\delta_0, \delta'\}$.

We may assume that $\mathcal{P}_0$ contains at least one minimal projection of each summand of C. Note that, without loss of generality, we may also assume that q commutes with each element in $\mathcal{S}$ and $\mathcal{H}$.

Without loss of generality (by omitting a possible error of $\delta/16$), we may also assume that each $u \in U(B) \cap \mathcal{P}$ has the form $quq \oplus (1-q)u(1-q)$, where $quq \in U(qBq)$ and $(1-q)u(1-q) \in U(C)$. We may further assume that $q \in B_1$ and $quq \in U(B_1)$. Let $\mathcal{U}' = \{quq\colon u \in U(B) \cap \mathcal{P}\}$ and let F be the subgroup of $U(qBq)$ generated by $\mathcal{U}'$. Let $\bar{F}$ be the image of F in $U(qBq)/CU(qBq)$. By 6.6(3), we write $\bar{F} = \bar{F} \cap U_0(qBq)/CU(qBq) \oplus \bar{F_0} \oplus \bar{F_1}$, where $\bar{F_0}$ is torsion and $\bar{F_1}$ is free. Note $\kappa(\bar{F_1}) \cong \bar{F_1}$. We may assume that $\mathcal{U}' = \mathcal{U}_0 \cup \mathcal{U}_1$ such that $\overline{\mathcal{U}_0}$ generates $\bar{F} \cap U_0(qBq)/CU(qBq) \oplus \bar{F_0}$ and $\overline{\mathcal{U}_1}$ generates $\overline{F_1}$. Note that modulo unitaries in $CU(qBq)$, we can always make this assumption and it will cost us no more than 8π in the estimation of the exponential length (see 6.9). Without loss of generality, we may also assume that $q \in B_1$ and $\mathcal{U}_0, \mathcal{U}_1 \subset qB_1q$. Note we also assume that $K_1(B_m) \to K_1(B_{m+1}) \to K_1(B)$ is injective.

Let $\mathcal{G}_1'$ be a finite subset which contains $\mathcal{S}$, $\mathcal{G}_0$ and $\mathcal{H}$ as well as $q, p_1, \ldots, p_n$ and a finite generating set of C_1'. It also contains $\mathcal{U}'$. Without loss of generality, we may further assume that $\mathcal{G}_1' \subset B_1$. Note we still have that q commutes with all elements in $\mathcal{G}_1'$. It follows from 10.3 (or 9.10) that there is a $\mathcal{G}_1'$-$\delta/4$-multiplicative contractive completely positive linear map $L_1 : B \to A$ such that

$$[L_1]|_{\mathcal{P}\cup\mathcal{P}_0} = \alpha^{-1}|_{\mathcal{P}\cup\mathcal{P}_0} \quad \text{and}$$
$$\sup_{\tau \in T(B)} \left\{ \left| \tau \circ L_1(a) - \xi^{-1}\big(Q(a)\big)(\tau) \right| \right\} < \sigma/2 \quad \text{for all } a \in \mathcal{H},$$

where $Q : A_{\mathrm{sa}} \to \mathit{Aff}(T(A))$ is the evaluation map.

We assume that $L_1^{\ddagger}$ is well defined on $\overline{F}$ $(\subset U(qBq)/CU(qBq))$ (see 6.2). Define $\mathbf{L}_1(A) \to \mathbf{R}_+$ exactly in the same way as $\mathbf{L}$ above. Let $\mathcal{F}_1$ be a finite subset of A. Let $\delta_1' > 0$, integer $n_1 > 0$, finite subsets $\mathcal{P}_1 \subset \mathbf{P}(A)$, $\mathcal{S}_1, \subset A$ (for A, $\mathbf{L}_1$, $\varepsilon/4$ and $\mathcal{F}_1$) be as required in Theorem 8.6. There are mutually orthogonal projections $q', p_1', p_2', \ldots, p_{n_1}'$ with $q' \lesssim p_i'$, and with $[p_i'] = [p_1']$ $(i = 1, 2, \ldots, n_1)$ and there is a C^*-subalgebra $C_2' \in \mathcal{I}$ with $1_{C_2'} = q'$ and there are unital S_1-$\delta_1'/4$-multiplicative contractive completely positive linear maps $h_0' : A \to q'Aq'$ and $h_1 : A \to C_2'$ such that $h_0(x) = q'xq'$ and

$$\left\| x - \big(h_0'(x) \oplus h_1'(x) \oplus \cdots \oplus h_1'(x)\big) \right\| < \delta_1'/16 \quad \text{for all } x \in S_1,$$

where h_1' is repeated n_1 times. We may assume that $L_1(\mathcal{S}) \subset \mathcal{S}_1$. Put $C_1 = M_{n_1}(C_2') \subset (1-q)A(1-q)$. Let $\mathcal{P}_{01}$, $\mathcal{G}_{01}$, $\mathcal{H}_1$, δ_{01}, and σ_{01} also be as required by 8.6. Let $\delta_1 = \min\{\delta_1', \delta_{01}\}$. We may assume that $\delta_1 < \delta/2$, $\sigma_1 < \sigma/4$ and $\mathcal{P}_{01}$ contains at least one minimal projection in each summand of C_1. Furthermore, without loss of generality, we may also assume that q' commutes with each element in $\mathcal{H}_1$ and $\mathcal{S}_1$, and that $[\mathcal{P}_1] \supset [L_1](\mathcal{P} \cup \mathcal{P}_0)$.

Without loss of generality, we may assume that each $u \in U(A) \cap \mathcal{P}_1$ has the form $q'uq' + (1-q')u(1-q')$, where $q'uq' \in U(q'Aq')$ and $(1-q')u(1-q') \in U((1-q')C_1(1-q'))$. Put $\mathcal{V}' = \{q'uq'\colon u \in U(A) \cap \mathcal{P}\}$. Let F' be the subgroup of $U(q'Aq')$ generated by $\mathcal{V}'$. By 6.6(3), we write $\overline{F'} = \overline{F'} \cap U_0(q'Aq')/CU(q'Aq') \oplus \overline{F_0'} \oplus \overline{F_1'}$, where $\overline{F_0'}$ is torsion and $\overline{F_1'}$ is free. Note $\kappa(\overline{F_1'}) \cong \overline{F_1'}$. We may also assume that $\bar{F}' \supset L_1^{\ddagger}(\bar{F})$. Since A is simple and separable, $K_1(q'Aq') = K_1(A)$. Without loss of generality, we may also assume that $\mathcal{V}' = \mathcal{V}_0 \cup \mathcal{V}_1$, where $\overline{\mathcal{V}_0}$ generates $\overline{F'} \cap U_0(q'Aq')/CU(q'Aq') \oplus \overline{F_0'}$ and $\overline{\mathcal{V}_1}$ generates $\overline{F_1'}$. Note again this assumption will cost us no more than 8π when we estimate the exponential length later (see 6.9).

Let $\mathcal{G}_2'$ be a finite subset which contains $\mathcal{S}_1$, $\mathcal{G}_{01}$, $\mathcal{H}_1$ and $L_1(\mathcal{G}_1')$ as well as $q', p_1', \ldots, p_{n_1}'$ and a finite generating set of C_2. It also contains $\mathcal{V}'$. Note q' commutes with all elements in $\mathcal{G}_2'$. It follows from 10.3 that there is a $\mathcal{G}_2'$-$\delta_1/2$-multiplicative contractive completely positive linear map $\Phi_1' : A \to B$ such that

$$[\Phi_1']|_{\mathcal{P}_1 \cup \mathcal{P}_{01}} = \alpha|_{\mathcal{P}_1 \cup \mathcal{P}_{01}} \quad \text{and}$$
$$\sup_{\tau \in T(A)} \{|\tau \circ \Phi_1'(a) - \xi(Q(a))(\tau)|\} < \sigma_1/4 \quad \text{for all } a \in \mathcal{H}_1 \cup L_1(\mathcal{H}).$$

We may also assume that $(\Phi_1')^{\ddagger}$ is well defined on $\overline{F'}$ and $(\Phi_1' \circ L_1)^{\ddagger}$ is well defined on $\overline{F}$. Without loss of generality (see [32, 6.2]), we may assume that the image of Φ_1' is contained in B_n.

Let $B_n' = qB_nq$. Since B is simple, it is known and easy to see that, by choosing a possibly large n, we may assume that the rank of q at each point is at least 6 (in B_n). We have assumed that $q \in B_n$. So B_n' is a corner of B_n. By the construction, we know that $[\Phi_1' \circ L_1](q)$ is equivalent to q. By replacing Φ_1' by $\mathrm{ad}w \circ \Phi_1'$ for some unitary w if necessary, we may assume that

$$\|\Phi_1' \circ L_1(q) - q\| < \delta/4.$$

Define $\Lambda(b) = aq[\Phi_1 \circ L_1(qbq)]qa$ (for $a = (q\Phi_1 \circ L_1(q)q)^{-1/2}$ and) for $b \in qBq$. Note that

$$\|\Lambda - \Phi_1' \circ L_1|_{qBq}\| < \delta/2.$$

Write $B_n = \bigoplus_{j=1}^m B_n(j)$, where each $B_n(j)$ has the form $C^{(j)}$ as described in 7.1. According to this direct sum decomposition, we may write $q = q_1 \oplus q_2 \oplus \cdots \oplus q_l$ with $0 \leqslant l \leqslant m$ and $q_j \neq 0$, $1 \leqslant j \leqslant l$. Choose an integer $N_1 > 0$ such that $N_1[q_j] \geqslant 3[1_{B_n(j)}]$ for $j \leqslant l$. Note that we may assume that q_j has rank at least 6. By applying an inner automorphism, we may assume that $\bigoplus_{j=1}^l B_n(j)$ is a hereditary C^*-subalgebra of $M_{N_1}(B_n')$. Since F_1 is finitely generated, with sufficiently large n, we obtain (see 6.2) a homomorphism $j : \bar{F}_1 \to U(qB_n'q)/CU(qB_n'q)$ such that $\psi_n^{\ddagger} \circ j = \mathrm{id}_{\bar{F}_1}$. Then (since $K_1(B_1) \to K_1(B_n) \to K_1(B)$ is injective),

$$\kappa_1 \circ \psi_n^{\ddagger} \circ (\Phi_1' \circ L_1)^{\ddagger}|_{\overline{F_1}} = \kappa_1 \circ (\psi_n)^{\ddagger} \circ j = (\kappa_1)|_{\overline{F_1}},$$

where $\kappa_1 : U(qBq)/CU(qBq) \to K_1(qBq)$ is the quotient map. Note that $K_1(qBq) = K_1(B)$. Let Δ_1 be $\delta(\varepsilon/4)$ as described in 7.5. We may assume that $\Delta_1 < \sigma_1/4$. To simplify notation, without loss of generality, we may assume that $\psi_n(q) = q$. By the assumption on B, we may write that $\psi_n|_{B_n'} = (\psi_n)_0 \oplus (\psi_n)_1$, where

(1) $\tau((\psi_n)_0(1_{B_n'})) < \Delta_1/2(N_1+1)^2$ for all $\tau \in T(B)$ and
(2) $(\psi_n)_0$ is homotopically trivial (but nonzero)

(see 10.1).

It follows from 7.5 that there is a homomorphism $h: B_n' \to e_0 B e_0$ such that:

(i) $[h] = [(\psi_n)_0]$ in $KL(B_n', B)$ and
(ii) $(\psi_n^\ddagger \circ j(\bar{w}))^{-1}(h \oplus (\psi_n)_1)^\ddagger(\Lambda^\ddagger(\bar{w})) = \overline{g_w}$, where $g_w \in U_0(qBq)$ and $\mathrm{cel}(g_w) < \varepsilon/4$ (in $U(qBq)$) for all $w \in \mathcal{U}_1$.

Define (we have assumed that $B_n \subset M_{N_1}(B_n')$)

$$h' = \big(\big(h \oplus (\psi_n)_1\big) \otimes \mathrm{id}_{M_{N_1}}\big)\big|_{\bigoplus_{j=1}^{l} B_n(j)}$$

and define $\Psi' = h' \oplus (\psi_n)|_{\bigoplus_{j=l+1}^{m} B_n(j)}$. Let $\Phi_1 = \Psi' \circ \Phi_1'$. It is clear that (since $\Delta_1 < \sigma_1/4$)

$$[\Phi_1]|_{\mathcal{P}_1 \cup \mathcal{P}_{01}} = \big[\Phi_1'\big]\big|_{\mathcal{P}_1 \cup \mathcal{P}_{01}} \quad \text{and} \quad \big|\tau \circ \Phi_1(a) - \tau \circ \Phi_1'(a)\big| < \sigma_1/2$$

for all $a \in A_{\mathrm{sa}}$ and $\tau \in T(B)$. For all $w \in \mathcal{U}_1$, we have, by (ii) above,

$$\mathrm{cel}\big(w^*\big(\Phi_1 \circ L_1(w)\big)\big) < 8\pi + \varepsilon/4 \quad \text{in } U(qBq).$$

For $w \in \mathcal{U}_0$, by 6.8, 6.10 and 6.9, we also have

$$\mathrm{cel}\big(w^*\big(\Phi_1 \circ L_1(w)\big)\big) < 2\,\mathrm{cel}(w) + \pi/64 \quad \big(\text{or } < 8\pi + 2\,\mathrm{cel}\big(w^k\big)/k + \pi/16\big) \quad \text{in } U(qBq)$$

(depending if $[w] = 0$ or $[w]$ has order k in $K_1(B)$). (Recall the definition of h_0 earlier in this proof in the next estimate.) Therefore (even after we add 8π for the decomposition assumption of $\overline{F}$)

$$\mathrm{cel}(\mathrm{id}_B\big(h_0(u)\big)^{-1}\big(\Phi_1 \circ L_1\big(h_0(u)\big)\big) < \mathbf{L}(u) \quad \text{in } U(qBq)$$

for all $u \in U(B) \cap \mathcal{P}_1$. Since we also have

$$[\mathrm{id}_B]|_{\mathcal{P} \cup \mathcal{P}_0} = [\Phi_1 \circ L_1]|_{\mathcal{P} \cup \mathcal{P}_0} \quad \text{and} \quad \sup_{\tau \in T(B)} \big\{\big|\tau(a) - \tau\big(\Phi_1 \circ L_1(a)\big)\big|\big\} < \sigma$$

for all $a \in \mathcal{H}$, by 8.6, we obtain a unitary $W \in U(B)$ such that

$$\mathrm{ad}\, W \circ \Phi_1 \circ L_1 \approx_{\varepsilon/2} \mathrm{id}_B \quad \text{on } \mathcal{F}.$$

Replacing Φ_1 by $\mathrm{ad}\, W \circ \Phi_1$, we may assume that

$$\Phi_1 \circ L_1 \approx_{\varepsilon/2} \mathrm{id}_B \quad \text{on } \mathcal{F}.$$

Now let $\mathcal{F}_2 \subset B$. We may assume that $\mathcal{F}_2 \subset B_{m_1'}$ ($m_1' > n$). Let $\delta_2' > 0$, integer $n_2 > 0$, finite subsets $\mathcal{P}_2 \subset \mathbf{P}(B)$, $\mathcal{S}_2, \subset B$, be as required by Theorem 8.6 (for B, $\mathbf{L}$, $\varepsilon/16$ and $\mathcal{F}_2$). There are mutually orthogonal projections q'', p_1'', p_2'', …, p_n'' with $q'' \lesssim p_i''$, and with $[p_i''] = [p_1'']$ ($i =$

$1, 2, \ldots, n$) and there is a C^*-subalgebra $C_3' \in \mathcal{I}$ with $1_{C_3'} = q''$, and there are unital $\mathcal{S}_2$-$\delta_2'/4$-multiplicative contractive completely positive linear maps $h_0'' : B \to q''Bq''$ and $h_1'' : B \to C_3'$ such that $h_0''(x) = q''xq''$ and

$$\left\| x - \left(h_0''(x) \oplus h_1''(x) \oplus \cdots \oplus h_1''(x)\right)\right\| < \delta_2'/4$$

for all $x \in \mathcal{S}_2$, where h_1'' is repeated n_2 times. We assume that $\mathcal{S}_2 \supset \Phi_1(\mathcal{S}_1)$. Put $C_2 = M_{n_2}(C_3') \subset (1-q'')B(1-q'')$. Let $\mathcal{P}_{02}$, $\mathcal{G}_{02}$, $\mathcal{H}_2$, $\delta_{02} > 0$ and $\sigma_2 > 0$ also be as required by 8.6. Let $\delta_2 = \min\{\delta_2', \delta_{02}\}$. We may assume that $\sigma_2 < \sigma_1/4$ and $\delta_2 < \delta_1/4$. We may also assume that $\mathcal{P}_{02}$ contains at least one minimal projection of each summand of C_2 and $[\mathcal{P}_2] \supset [\Phi_1(\mathcal{P}_1 \cup \mathcal{P}_{01})]$. Furthermore, we may assume that each $u \in U(B) \cap \mathcal{P}_2$ has the form $q''uq'' \oplus (1-q'')u(1-q'')$, where $q''uq'' \in U(q''Bq'')$ and $(1-q'')u(1-q'') \in U(C_2)$. Put $\mathcal{W} = \{q''uq'': u \in U(B) \cap \mathcal{P}_2\}$. Let F'' be the subgroup generated by $\mathcal{W}$. Write $\overline{F''} = \overline{F''} \cap U_0(q''Bq'')/CU(q''Bq'') \oplus \overline{F_0''} \oplus \overline{F_1''}$, where $\overline{F_0''}$ is torsion and $\overline{F_1''}$ is free. We may also assume that $\overline{F}'' \supset \Phi_1^{\ddagger}(\overline{F}')$. We may further assume, without loss of generality, that $\mathcal{W} = \mathcal{W}_0 \cup \mathcal{W}_1$, where $\mathcal{W}_0$ generates $\overline{F''} \cap U_0(q''Bq'')/CU(q''Bq'') \oplus \overline{F_0''}$ and $\mathcal{W}_1$ generates $\overline{F_1''}$.

Let $\mathcal{G}_3'$ be a finite subset which contains $\mathcal{S}_2$, $\mathcal{G}_{02}$, q'', $p_1'', \ldots, p_{n_1}''$, $\mathcal{H}_2$, $\Phi_1(\mathcal{G}_2'')$, a generating set of C_2 and $\mathcal{W}$. Without loss of generality, we may assume (see [31, 6.2]) that $\Phi_1(A) \subset B_m$ ($m > m_1 > n$). By choosing a larger m, we may assume that there is a contractive completely positive linear map $J : B \to B_m$ such that

$$\left\| J(a) - \mathrm{id}(a)\right\| < \delta_2/8 \quad \text{for all } a \in \mathcal{G}_3'.$$

There is a projection $\tilde{q}' \in B_m$ such that

$$\left\| \Phi_1(q') - \tilde{q}'\right\| < \delta_2/2.$$

We may write $B_m = \bigoplus_{j=1}^s B_m(j)$. As above, by choosing a possibly larger m, we may also assume that $\tilde{q}'$ has rank at least 6 at each point. We write that $\tilde{q}' = q_1' \oplus q_2' \oplus \cdots \oplus q_l'$ according to the direct sum decomposition ($q_j' \neq 0$ for $1 \leqslant j \leqslant l$ and $l \leqslant s$). Suppose that $N_2 > 0$ is an integer such that $N_2[q_j'] > 3[1_{B_m(j)}]$ for $1 \leqslant j \leqslant l$. Set $B_m' = \tilde{q}' B_m \tilde{q}'$. Note $\Phi_1^{\ddagger}$ is injective on $\bar{F}_1'$. We may further assume that $\mathcal{G}_3'$ contains $q_1', q_2', \ldots, q_l'$ and a generating set of B_m' and B_m.

Now let $L_2' : B \to A$ be a $\mathcal{G}_3'$-$\delta_2/16(N_2+1)^2$-multiplicative contractive completely positive linear map (10.3) such that

$$[L_2']|_{\mathcal{P}_2} = \alpha^{-1}|_{\mathcal{P}_2 \cup \mathcal{P}_{02}} \quad \text{and}$$

$$\sup_{\tau \in T(A)} \left\{\left|\tau \circ L_2'(a) - \xi^{-1}\left(Q(a)\right)(\tau)\right|\right\} < \sigma_2/4 \quad \text{for all } a \in \mathcal{H}_2 \cup \Phi_1(\mathcal{H}_1).$$

We may assume that $(L_2')^{\ddagger}$ is well defined on $\overline{F''}$. Suppose that $e \in A$ is a projection such that

$$\left\| L_2' \circ \Phi_1(q') - e\right\| < \delta_2/4.$$

Since $q' \in \mathcal{P}_1$, $[e] = [q']$ in $K_0(A)$. Thus, by replacing L_2' by $\mathrm{ad}\,u' \circ L_2'$ for some unitary $u' \in A$, we may assume that $e = q'$. Without loss of generality, to simplify notation, we may assume $L_2' \circ \Phi_1(q') = q'$. Note $\overline{F_1'}$ is free and $(\phi_{m,M})_{*1}$ ($M > m$) is injective. We compute that

$$\alpha^{-1} \circ (\psi_m)_{*1} \circ \kappa_1' \circ (\Phi_1)^{\ddagger}(g) = \kappa_1(g)$$

for all $g \in \overline{F_1'}$, where $\kappa_1' : U(B_m')/CU(B_m') \to K_1(B_m')$ and $\kappa_1 : U(\tilde{q}'B\tilde{q}')/CU(\tilde{q}'B\tilde{q}') \to K_1(\tilde{q}'B\tilde{q}')$ are the quotient maps. Note that $K_1(\tilde{q}'B\tilde{q}') = K_1(B)$. With $\Phi_1^{\ddagger}$ playing the role of L and $\alpha^{-1} \circ (\psi_m)_{*1}$ playing the role of α in 7.3, by applying 7.3, we obtain a homomorphism $\beta : U(B_m')/CU(B_m') \to U(q'Aq')/CU(q'Aq')$ with $\beta(U_0(B_m')/CU(B_m')) \subset U_0(q'Aq')/CU(q'Aq')$ such that

$$\beta \circ \left(\Phi_1^{\ddagger}\right)(\bar{w}) = \bar{w}$$

for all $\bar{w} \in \overline{F_1'}$. Let $\Delta_2' = \delta(\varepsilon/16)$ be as described in 7.4. It follows from the assumption on B that there is $M > m$ such that $\phi_{m,M} = \phi_{m,M}^{(0)} \oplus \phi_{n,M}^{(1)} : B_m \to B_M$ such that $\phi_{m,M}^{(0)}$ is (nonzero) homotopically trivial and $\tau(\psi_M \circ \phi_{n,M}^{(0)}(1_{B_m'})) < \Delta_2'/4(N_2+1)^2$ for all $\tau \in T(B)$. To simplify notation, without loss of generality, we may also assume that $e_0' = L_2' \circ \psi_M \circ \phi_{m,M}^{(0)}(1_{B_m'})$ and $e_1' = L_2' \circ \psi_M \circ \phi_{m,M}^{(0)}(1_{B_m'})$ are mutually orthogonal projections (see 8.2(ii)). Note that $\psi_M \circ \phi_{m,M} = \psi_m$. It follows from 7.4 (by the choice of Δ_2' and with β playing the role of α, $L_2' \circ \psi_M \circ \phi_{m,M}^{(0)}$ playing the role of ϕ_0 and $L_2' \circ \psi_M \circ \phi_{m,M}^{(1)}$ playing the role of ϕ_1 in 7.4), we obtain a homomorphism $\Phi' : B_m' \to e_0'Ae_0'$ such that:

(i′) Φ' is homotopically trivial, $\Phi'_{*0} = [L_2'] \circ (\psi_M \circ \phi_{m,M}^{(0)})_{*0}|_{K_0(B_m')}$ and

(ii′) $(\beta(\Phi_1^{\ddagger}(\bar{w}))^{-1})(\Phi' \oplus (L_2' \circ \psi_M \circ \phi_{m,M}^{(1)}))^{\ddagger}(\Phi_1^{\ddagger}(\bar{w})) = \overline{g_w}$, where $g_w \in U_0(q'Aq')$ and $\mathrm{cel}(g_w) < \varepsilon/4$ (in $U(q'Aq')$) for all $w \in \mathcal{V}_1$. As in the construction of Ψ', we obtain a homomorphism $\tilde{\Phi}' : B_m \to q'Aq'$ such that $\tilde{\Phi}'$ is homotopically trivial, $\tilde{\Phi}'_{*0} = [L_2'] \circ (\psi_M \circ \phi_{m,M})_{*0}$ and $\tilde{\Phi}'|_{B_m'} = \Phi'$.

Define $L_2 = (\tilde{\Phi}' \oplus L_2' \circ \psi_M \circ \phi_{n,M}^{(1)}) \circ J$. It is clear that

$$[L_2]|_{\mathcal{P}_2 \cup \mathcal{P}_{02}} = \left[L_2'\right]\big|_{\mathcal{P}_2 \cup \mathcal{P}_{02}} = \alpha^{-1}\big|_{\mathcal{P}_2 \cup \mathcal{P}_{02}}.$$

Given the choice of Δ_2, we also have

$$\left|\tau \circ L_2(a) - \tau\left(L_2'(a)\right)\right| < \sigma_2/4$$

for all $a \in A_{\mathrm{sa}}$ and $\tau \in T(A)$. In particular,

$$\sup_{\tau \in T(A)} \left\{\left|\tau \circ L_2 \circ \Phi_1(a) - \tau(a)\right|\right\} < \sigma_1/2$$

for all $a \in \mathcal{H}_1$. Since $\beta \circ \Phi_1^{\ddagger}(\bar{w}) = \bar{w}$ for all $w \in \mathcal{V}_1$, by (ii′) and 6.9, we have

$$\mathrm{cel}\left(\mathrm{id}_A\left(h_0'(w^*)\right)L_2\left(\Phi_1\left(h_0'(w)\right)\right)\right) < 8\pi + \mathrm{cel}(g_w) + \varepsilon/4 < 8\pi + \varepsilon/2 \quad \text{in } U(q'Aq')$$

for all $w \in \mathcal{V}_1$. We also have, by 6.8–6.10,

$$\operatorname{cel}\bigl(\mathrm{id}_A\bigl(h_0'(w^*)\bigr)L_2\bigl(\Phi_1\bigl(h_0'(w)\bigr)\bigr)\bigr) < 2\operatorname{cel}(w) + \pi/16$$
$$\bigl(\text{or } < 8\pi + 2\operatorname{cel}\bigl(w^k\bigr)/k + \pi/16\bigr) \quad \text{in } U(q'Aq')$$

for all $w \in \mathcal{V}_0$ (depends on if $[w] = 0$ in $K_1(A)$ or $[w]$ has order k). It follows that (by adding 8π)

$$\operatorname{cel}\bigl(\mathrm{id}\bigl(h_0'(u^*)\bigr)L_2\bigl(\Phi_1\bigl(h_0'(u)\bigr)\bigr)\bigr) < \mathbf{L}(u)$$

for all $u \in U(A) \cap \mathcal{P}_2$ (in $q'Aq'$).

By applying 8.6, we obtain a unitary $Z \in U(A)$ such that

$$\operatorname{ad} Z \circ L_2 \circ \Phi_1(a) \approx_{\varepsilon/16} \mathrm{id}_A \quad \text{on } \mathcal{F}_1.$$

Therefore, by replacing L_2 by $\operatorname{ad} Z \circ L_2$, we obtain the following "approximate intertwining" diagram:

$$\begin{tikzcd} B \arrow[r, "\mathrm{id}_B"'] \arrow[d, "L_1"] & B \arrow[d, "L_2"] \\ A \arrow[ur, "\Phi_1"'] \arrow[r, "\mathrm{id}_A"'] & A. \end{tikzcd}$$

Since this process continues, we see that L_1 is recursively $\mathcal{F}$-invertible (and Φ_1 is recursively $\mathcal{F}_1$-invertible—see [32, 3.6]). It follows from an argument of Elliott (see [32, Theorem 3.6], for example) that A is isomorphic to B. $\square$

10.5. Remark. If $K_1(A)$ and $K_1(B)$ are torsion groups, then one can use the "uniqueness theorem" 8.7. Since we do not need to control exponential length in this case, Section 7 is not needed. Furthermore, we do not need to assume B is AH. Consequently, we do not need to assume the condition on the $\partial_{\mathrm{e}}(S_{\mathrm{u}}(K_0(A))$ either. The whole proof is much shorter.

Since simple AH-algebras with very slow dimension growth have tracial rank one or zero, we have the following.

10.6. Theorem. *(See [20] and [23].) Let A and B be two unital simple AH-algebras with very slow dimension growth and with torsion $K_1(A)$. Then A is isomorphic to B if and only if*

$$\bigl(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A)\bigr) \cong \bigl(K_0(B), K_0(B)_+, [1_B], K_1(B), T(B)\bigr)$$

in the sense of 10.2.

Note that 2.5 only uses [23, Section 4]. Thus 10.6 does not use [23, Sections 5 and 6] nor it uses [20].

10.7. Let C be a stably finite non-unital C^*-algebra with an approximate identity consisting of projections $\{e_n\}$. Let $T(C)$ denotes the set of traces τ on C such that $\sup_n \tau(e_n) = 1$. We refer to these traces as tracial states on C, and to $T(C)$ the tracial state space of C. Note that each tracial state extends to a tracial state on $\tilde{C}$. Therefore $T(\tilde{C})$ is the set of convex combinations of $\tau \in T(C)$ and the tracial state which vanishes on C. We also denote by $S_{u'}(K_0(C))$ the set of those order preserving homomorphisms from $K_0(C)$ to $\mathbb{R}$ such that $\sup_n s([e_n]) = 1$. Then each element in $S_u(K_0(\tilde{C}))$ is the convex combination of $s \in S_{u'}(K_0(C))$ and the state which vanishes on $j_*(K_0(C))$, where $j : C \to \tilde{C}$ is the embedding.

10.8. Lemma. *Let A be a unital separable simple C^*-algebra with $TR(A) \leqslant 1$. Then there is a C^*-algebra $C = \lim_{n\to\infty}(C_n, \phi_n)$, where $C_n \in \mathcal{I}$, satisfying the following*:

(i) *each C_n is a C^*-subalgebra of A and $\{\phi_{n,\infty}(1_{C_n})\}$ forms an approximate identity for C*;
(ii) *there is a sequence of contractive completely positive linear maps $L_n : A \to C$ such that*

$$\lim_{n\to\infty} \left\| L_n(ab) - L_n(a)L_n(b) \right\| = 0, \quad a, b \in A;$$

(iii) *there is an affine continuous (face-preserving) isomorphism $r^\sharp : T(A) \to T(C)$ such that*

$$r^\sharp(\tau)\big(\phi_{n,\infty}(b)\big) = \lim_{k\to\infty} \tau\big(\phi_{n,k}(b)\big) \quad \textit{for all } b \in C_n \textit{ and } \tau \in T(A);$$

(iv) *there is an affine continuous (face-preserving) isomorphism $r_\sharp : S_{u'}(K_0(C)) \to S_u(K_0(A))$ such that*

$$r_\sharp(s)\big([p]\big) = \lim_{n\to\infty} \tau_s\big(L_n(p)\big) \quad \textit{for all } s \in S_{u'}\big(K_0(C)\big) \textit{ and projection } p \in A,$$

where τ_s is the trace which induces s.

Proof. Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \subset \cdots$ be a sequence of finite subsets of A such that $\bigcup_n \mathcal{F}_n$ is dense in A. Since $TR(A) \leqslant 1$, there is a C^*-subalgebra $C_1 \subset A$ with $C_1 \in \mathcal{I}$ and $1_{C_1} = p_1$ such that:

(1′) $\|ap_1 - p_1 a\| < 1/2$ for all $a \in \mathcal{F}_1$,
(2′) $\operatorname{dist}(p_1 a p_1, C_1) < 1/2$ for all $a \in \mathcal{F}_1$,
(3′) $\tau(1 - p_1) < 1/4$ for all $\tau \in T(A)$.

Let $1 > \eta_1 > 0$. By (3′) and 4.7, there is a projection $e_{(1,1)} \leqslant p_1$ such that $e_{(1,1)}$ is equivalent to $1 - p_1$. Since $\tau(p_1 - e_{(1,1)}) > 1/2 > \tau(1 - p_1)$ for all $\tau \in T(A)$, by 4.7 again, we obtain mutually orthogonal projections $e_{(1,1)}, e_{(1,2)}$ such that $e_{(1,i)} \leqslant p_1$, $[e_{(1,1)}] = [e_{(1,2)}] \geqslant [1 - p_1]$. There are $x_{(1,1)}, x_{(1,2)} \in A$ such that $x_{(1,i)}^* x_{(1,i)} \geqslant 1 - p_1$ and $x_{(1,i)} x_{(1,i)}^* = e_{(1,i)}$. Let $\mathcal{G}_1'$ be a finite set of generators of C_1 and $\mathcal{G}_2 = \mathcal{F}_2 \cup \mathcal{G}_1' \cup \{x_{(1,i)}, x_{(1,i)}^*, e_{(1,i)}\colon 1 \leqslant i \leqslant 2\}$. There is a C^*-subalgebra $C_2 \subset A$ with $C_2 \in \mathcal{I}$ and $1_{C_2} = p_2$ such that:

(1″) $\|ap_2 - p_2 a\| < \eta_1/4$ for all $a \in \mathcal{G}_2$,
(2″) $\operatorname{dist}(p_2 a p_2, C_2) < \eta_1/4$ for all $a \in \mathcal{G}_2$, and
(3″) $\tau(1 - p_2) < 1/8$ for all $\tau \in T(A)$.

By 2.1(iii), with sufficiently small η_1, there is a homomorphism $\phi_1 : C_1 \to C_2$ such that

$$\left\|\phi_1(b) - p_2 b p_2\right\| < 1/4 \quad \text{for all } b \in \mathcal{F}_1 \cup \mathcal{G}_1'.$$

Put $q_2 = \phi_1(1_{C_1})$. With sufficiently small η_1, since $x_{(1,i)} \in \mathcal{G}_2$, we may also assume that $2[p_2 - q_2] \leqslant [q_2]$ in $K_0(C_2)$. Note that $q_2 \leqslant p_2$.

We continue in this fashion. Suppose that $C_n \subset A$ is a unital C^*-subalgebra which is in $\mathcal{I}$ has been constructed. If $\tau(1-p_n) < 1/2^{n+1}$ for all $\tau \in T(A)$, there are partial isometries $x_{(n,i)} \in A$ such that $p_n = 1_{c_n}$, $x_{(n,i)}^* x_{(n,i)} \geqslant 1 - p_n$, $x_{(n,i)} x_{(n,i)}^* = e_{(n,i)} \leqslant p_n$, $e_{(n,i)} e_{(n,j)} = 0$ if $i \neq j$ and $[e_{(n,i)}] = [e_{(n,1)}] \geqslant [1-p_n]$, $1 \leqslant i \leqslant 2^n$. Let $\mathcal{G}_n'$ be a finite set which contains a set of generators of C_n, $\phi_{i,n}(\mathcal{G}_i)$ and $\phi_{i,n}(p_i)$, $i = 1, 2, \ldots, n-1$, where $\phi_{i,n} = \phi_{n-1} \circ \phi_{n-2} \circ \cdots \circ \phi_i$. (Note that $C_i \subset A$.) Let $\mathcal{G}_{n+1} = \mathcal{F}_{n+1} \cup \mathcal{G}_n' \cup \{e_{(n,i)}, x_{(n,i)}, x_{(n,i)}^* \colon 1 \leqslant i \leqslant 2^n\}$. Let $1 > \eta_{n+1} > 0$ be a positive number to be determined (but it depends only on C_n and $\mathcal{F}_n \cup \mathcal{G}_n'$). Since A has tracial topological rank one, there exist a C^*-subalgebra $C_{n+1} \subset A$ with $C_{n+1} \in \mathcal{I}$ and a projection p_{n+1} with $1_{C_{n+1}} = p_{n+1}$ such that:

(1) $\|a p_{n+1} - p_{n+1} a\| < \eta_{n+1}/2^{n+1}$ for all $a \in \mathcal{G}_{n+1}$,
(2) $\operatorname{dist}(p_{n+1} a p_{n+1}, C_{n+1}) < \eta_{n+1}/2^{n+1}$ for all $a \in \mathcal{G}_{n+1}$ and
(3) $\tau(1 - p_{n+1}) < 1/2^{n+2}$ for all $\tau \in T(A)$.

We choose η_{n+1} so small that there exist a homomorphisms $\phi_n : C_n \to C_{n+1}$ (by 2.1(iii)) such that

$$\left\|\phi_n(b) - p_{n+1} b p_{n+1}\right\| < 1/2^{n+1} \quad \text{for all } b \in \mathcal{F}_n \cup \mathcal{G}_n'. \tag{e1}$$

Put $q_{n+1} = \phi_n(p_n)$. It is useful to note that $q_{n+1} \leqslant p_{n+1}$. Since $x_{(n,i)} \in \mathcal{G}_{n+1}$, we may further assume that

(4) $2^n[p_{n+1} - q_{n+1}] \leqslant [q_{n+1}]$ in $K_0(C_{n+1})$.

Set $C = \lim_{n\to\infty}(C_n, \phi_n)$. Since each C_n is nuclear C^*-subalgebra of A, there is a contractive completely positive linear map $L_n' : A \to C_n$ (see, for example, [34, 2.3.13]) such that

$$\lim_{n\to\infty} \left\|L_n'(a) - p_n a p_n\right\| = 0$$

for all $a \in A$. Note that, by (1),

$$\lim_{n\to\infty} \left\|L_n'(ab) - L_n'(a) L_n'(b)\right\| = 0 \quad \text{for all } a, b \in A.$$

Define $L_n = \phi_{n,\infty} \circ L_n'$. It is clear that L_n satisfies (ii). Put $\phi_{n,n+1} = \phi_n$ and for $k > n+1$, $\phi_{n,k} = \phi_{k-1} \circ \cdots \circ \phi_n$. Define $r^\sharp : T(A) \to T(C)$ as follows. For each $b \in C_n$, define

$$r^\sharp(\tau)\big(\phi_{n,\infty}(b)\big) = \lim_{k\to\infty} \tau\big(\phi_{n,k}(b)\big) \quad \text{for } \tau \in T(A).$$

Note that $\phi_{n,k}(b) \in C_k \subset A$. We will show that the right-hand side above converges. Since we may replace b by $\phi_{n,k}(b)$ (replacing n by a larger integer if necessary), without loss of generality, we may also assume that $b \in \mathcal{F}_n$. From (e1), one obtains that

$$\left\|\phi_{n,k+j+1}(b) - p_{k+j+2}\phi_{n,k+j}(b)p_{k+j+2}\right\| < 1/2^{k+j+2}. \tag{e2}$$

On the other hand, for any integer $k \geqslant 0$,

$$\begin{aligned}
&\left|\tau\left(p_{k+j+2}\phi_{n,k+j}(b)p_{k+j+2}\right) - \tau\left(\phi_{n,k+j}(b)\right)\right| \\
&\quad\leqslant \left|\tau\left((1-p_{k+j+2})\phi_{n,k+j}(b)(1-p_{k+j+2})\right)\right| + \left|\tau\left(p_{k+j+2}\phi_{n,k+j}(b)(1-p_{k+j+2})\right)\right| \\
&\qquad + \left|\tau\left((1-p_{k+j+2})\phi_{n,k+j}(b)p_{k+j+2}\right)\right| \\
&\quad< 3\|b\|\tau(1-p_{k+j+2}) \leqslant 3\|b\|/2^{k+j+2}.
\end{aligned} \tag{e3}$$

It follows from (e2) and (e3) that

$$\left|\tau\left(\phi_{n,k+j+1}(b)\right) - \tau\left(\phi_{n,k+j}(b)\right)\right| < 1/2^{k+j+2} + 3\|b\|/2^{k+j+2}.$$

Therefore, for any $m \geqslant 1$,

$$\left|\tau\left(\phi_{n,k}(b)\right) - \tau\left(\phi_{n,k+m}(b)\right)\right| < \sum_{j=0}^{m} 1/2^{k+j+2} + 3\|b\|\sum_{j=0}^{m} 1/2^{k+j+2} \to 0,$$

as $k \to \infty$. We conclude that $\lim_{k\to\infty}\tau(\phi_{n,k}(b))$ converges. To see $r^\sharp$ is well defined, we let $c \in C_m$ so that $\phi_{m,\infty}(c) = \phi_{n,\infty}(b)$. Then, for any $\varepsilon > 0$, there exists $N > \max\{n, m\}$ such that

$$\left\|\phi_{n,k}(b) - \phi_{m,k}(c)\right\| < \varepsilon \quad (\text{in } C_k)$$

for all $k \geqslant N$. It follows that ($C_k \subset A$)

$$\left|\tau\left(\phi_{n,k}(b)\right) - \tau\left(\phi_{m,k}(c)\right)\right| < \varepsilon$$

for all $\tau \in T(A)$ and $k \geqslant N$. It follows that $r^\sharp$ is well defined on $\bigcup_{n=1}^\infty \phi_{n,\infty}(C_n)$. Since $|\tau(\phi_{n,k}(b))| \leqslant \|\phi_{n,k}(b)\|$, $r^\sharp(\tau)$ is bounded linear functional on $\bigcup_{n=1}^\infty \phi_{n,\infty}(C_n)$. It defines (uniquely) a bounded linear functional on C. One then easily sees that $r^\sharp(\tau)$ is a state. Moreover, one checks that it is a tracial state. Thus $r^\sharp$ is well defined. It is then easy to see that $r^\sharp$ is an affine continuous map. Define $r^{\sharp\,-1} : T(C) \to T(A)$ by

$$r^{\sharp\,-1}(t)(a) = \lim_{n\to\infty} t\left(L_n(a)\right) = \lim_{n\to\infty} t\left(\phi_{n,\infty}\left(L_n'(a)\right)\right) \quad \text{for all } t \in T(C) \text{ and } a \in A.$$

To justify the definition, we first need to show that $\lim_{n\to\infty} t(\phi_{n,\infty}(L_n'(a)))$ exists. Let $a \in \mathcal{F}_n$ and $k > 0$. Define

$$b_{n,k,1}(a) = (p_{n+k+2} - q_{n+k+2})L'_{n+k+2}(a)(p_{n+k+2} - q_{n+k+2}),$$
$$b_{n,k,2}(a) = (p_{n+k+2} - q_{n+k+2})L'_{n+k+2}(a)q_{n+k+2} \quad \text{and}$$
$$b_{n,k,3}(a) = q_{n+k+2}L'_{n+k+2}(a)(p_{n+k+2} - q_{n+k+2})$$

and define

$$b_{n,k,1}(a)' = p_{n+k+2}(1 - p_{n+k+1})L'_{n+k+2}(a)(1 - p_{n+k+1})p_{n+k+2},$$
$$b_{n,k,2}(a)' = p_{n+k+2}(1 - p_{n+k+1})L'_{n+k+2}(a)p_{n+k+1}p_{n+k+2} \quad \text{and}$$
$$b_{n,k,3}(a)' = p_{n+k+2}p_{n+k+1}L'_{n+k+2}(a)(1 - p_{n+k+1})p_{n+k+2}.$$

Note that $b_{n,k,i}(a) \in C_{n+k+2}$, $i = 1, 2, 3$. By (e1) and (1) above, in A,

$$\big\|\big[\phi_{n+1} \circ L'_{n+k+1}(a) - L'_{n+k+2}(a)\big] - \big[b_{n,k,1}(a)' + b_{n,k,2}(a)' + b_{n,k,3}(a)'\big]\big\| < 5/2^{n+k+2} + 5\eta_{n+k+2}/2^{n+k+2}. \tag{e4}$$

We also estimate:

$$\big\|\big(b_{n,k,1}(a)' + b_{n,k,2}(a)' + b_{n,k,3}(a)'\big) - \big(b_{n,k,1}(a) + b_{n,k,2}(a) + b_{n,k,3}(a)\big)\big\| < 3/2^{n+k+2}.$$

It follows that (in C_{n+k+2})

$$\big\|\big(\phi_{n+1} \circ L'_{n+k+1}(a) - L'_{n+k+2}(a)\big) - \big(b_{n,k,1}(a) + b_{n,k,2}(a) + b_{n,k,3}(a)\big)\big\| < 8/2^{n+k+2} + 5\eta_{n+k+2}/2^{n+k+2}. \tag{e5}$$

By (e5),

$$\big\|\big(L_{n+k+1}(a) - L_{n+k+2}(a)\big) - \phi_{n+k+2,\infty}\big(b_{n,k,1}(a) + b_{n,k,2}(a) + b_{n,k,3}(a)\big)\big\| < 1/2^{n+k-1} + 5\eta_{n+k+2}/2^{n+k+2}. \tag{e6}$$

By (4), in $K_0(C)$,

$$2^{n+k+1}\big[\phi_{n+k+2,\infty}(p_{n+k+2} - q_{n+k+2})\big] \leqslant \big[\phi_{n+k+2,\infty}(q_{n+k+2})\big].$$

It follows that, for any $t \in T(C)$,

$$t\big(\phi_{n+k+2,\infty}(p_{n+k+2} - q_{n+k+2})\big) < 1/2^{n+k+1}.$$

From this, we estimate that

$$t\big(\phi_{n+k+2,\infty}\big(b_{n,k,i}(a)\big)\big) \leqslant \|a\|/2^{n+k+1}, \quad i = 1, 2, 3$$

for all $t \in T(C)$. Combing this with (e6), we have

$$\left|t\left(\left(L_{n+k+1}(a)\right)-t\left(L_{n+k+2}(a)\right)\right)\right|<1/2^{n+k-1}+5\eta_{n+k+2}/2^{n+k+2}+3\|a\|/2^{n+k+1}.$$

Hence

$$\left|t\left(L_{n+1}(a)\right)-t\left(L_{n+m}(a)\right)\right|<\sum_{k=0}^{m}\left(1/2^{n+k-1}+5\eta_{n+k+2}/2^{n+k+2}+3\|a\|/2^{n+k+1}\right)\to 0$$

as $n\to\infty$. This proves that $\lim_{n\to\infty} t(\phi_{n,\infty}(L_n'(a))$ exists. Then one shows that $r^{\sharp-1}(t)$ is well defined. By (ii), which we have shown, $r^{\sharp-1}(t)$ is a trace on A. It is then clear that $r^{\sharp-1}$ is an affine continuous map. It should be noted that even if $a\in C_m$ (for $m<n$), $L_n'(a)\in C_n$.

Now let $\tau\in T(A)$ and $a\in A$. To show that $(r^{\sharp-1}\circ r^{\sharp})(\tau)(a)=\tau(a)$, we note that

$$\left(r^{\sharp-1}\circ r^{\sharp}\right)(\tau)(a)=\lim_{n\to\infty} r^{\sharp}(\tau)\left(\phi_{n,\infty}\left(L_n'(a)\right)\right)=\lim_{n\to\infty}\left(\lim_{k\to\infty}\tau\left(\phi_{n,k}\left(L_n'(a)\right)\right)\right)$$

for all $a\in A$ and $\tau\in T(A)$. Let $\varepsilon>0$. Without loss of generality, we may assume that $a\in\mathcal{F}_n$ for some integer $n>0$. Moreover, with sufficiently large n, we may assume that $1/2^n<\varepsilon/8$ and

$$\left\|L_n'(a)-p_nap_n\right\|<\varepsilon/4.$$

One estimates, by (e1) (with $k>n$),

$$\left\|\phi_{n,k}\left(L_n'(a)\right)-p_kp_{k-1}\cdots p_{n+1}p_nap_np_{n+1}\cdots p_{k-1}p_k\right\|<\sum_{j=1}^{k-n}1/2^{n+j}+\varepsilon/4<\varepsilon/2.$$

By (3) and as in (e3), one has

$$\begin{aligned}&\left|\tau(p_kp_{k-1}\cdots p_{n+1}p_nap_np_{n+1}\cdots p_{k-1}p_k)-\tau(a)\right|\\&\quad<3\|a\|\sum_{j=n}^{k-n}1/2^{n+2}<3\|a\|\varepsilon/8\quad\text{for all }\tau\in T(A).\end{aligned}$$

It follows that

$$\left|\tau\left(\phi_{n,k}\left(L_n'(a)\right)\right)-\tau(a)\right|<3\|a\|\varepsilon/8+\varepsilon/2\quad\text{for all }\tau\in T(A)$$

if $k>n$. Therefore

(5) $\tau(a)=\lim_{n\to\infty}(\lim_{k\to\infty}\tau(\phi_{n,k}(L_n'(a))))$ for all $a\in A$ and $\tau\in T(A)$.

This also proves $r^{\sharp-1}\circ r^{\sharp}(\tau)(a)=\tau(a)$ for all $a\in A$ and $\tau\in T(A)$. Therefore $r^{\sharp-1}\circ r^{\sharp}=\mathrm{id}_{T(A)}$.

Suppose that $t\in T(C)$ and $b\in C_n$. Then

$$r^{\sharp}\circ r^{\sharp-1}(t)\left(\phi_{n,\infty}(b)\right)=\lim_{k\to\infty} r^{\sharp-1}(t)\left(\phi_{n,k}(b)\right)=\lim_{k\to\infty}\left(\lim_{m\to\infty} t\left(\phi_{m,\infty}\left(L_m'\left(\phi_{n,k}(b)\right)\right)\right)\right).$$

Fix $\varepsilon > 0$. Choose $k > n$ such that $1/2^k < \varepsilon/32$. We may assume that $\|b\| \leqslant 1$. For any $m > k$, put $r_j = \phi_{j,m}(p_j)$, $j = k, \ldots, m-1$. Since $\phi_j(p_j) \leqslant p_{j+1}$, $r_j \leqslant r_{j+1}$. By choosing a larger k, applying (1) and (2) above, we may assume that there is $c_1 \in A$ such that (we view $\phi_{n,k}(b) \in C_k \subset A$)

$$r_j c_1 = c_1 r_j, \qquad k+1 \leqslant j \leqslant m-1, \quad \text{and} \quad \left\|c_1 - \phi_{n,k}(b)\right\| < \varepsilon/8.$$

We also have

$$\left\|\phi_{n,m}(b) - p_m r_{m-1} \cdots r_k \phi_{n,k}(b) r_k \cdots r_{m-1} p_m\right\| < \sum_{j=1}^{m-k} 2^{k+j} < \varepsilon/8.$$

Put $c_2 = p_m r_{m-1} \cdots r_k c_1$. It then follows that

$$c_3 = L'_m(c_1) - c_2 \leqslant 2(p_m - r_k).$$

Since each C_j has stable rank one, by (4), there are $y_i \in C_m$ such that $y_i^* y_i = p_m - r_k$ and $y_i y_i^*$ $(1 \leqslant i \leqslant 2^m)$ are mutually orthogonal. Let $z_i = \phi_{m,\infty}(y_i)$, $i = 1, 2, \ldots, 2^m$. Then $z_i^* z_i = \phi_{m,\infty}(p_m - r_k)$ and $z_i z_i^*$ $(1 \leqslant i \leqslant 2^m)$ are mutually orthogonal. It follows that

$$t\big(\phi_{m,\infty}(c_3)\big) \leqslant 2\big(1/2^m\big) < \varepsilon/8$$

for all $t \in T(C)$. On the other hand, from the above estimates,

$$\begin{aligned}
&\left\|\left[\phi_{m,\infty}\big(L'_m\big(\phi_{n,k}(b)\big)\big) - \phi_{n,\infty}(b)\right] - \phi_{m,\infty}(c_3)\right\| \\
&\quad \leqslant \left\|\left[L'_m\big(\phi_{n,k}(b)\big) - \phi_{n,m}(b)\right] - c_3\right\| \\
&\quad \leqslant \left\|L'_m\big(\phi_{n,k}(b)\big) - L'_m(c_1)\right\| + \left\|\phi_{n,m}(b) - c_2\right\| + \left\|\big(L'_m(c_1) - c_2\big) - c_3\right\| \\
&\quad \leqslant \varepsilon/8 + (\varepsilon/8 + \varepsilon/8) + 0 = 3\varepsilon/8.
\end{aligned}$$

It follows that

$$\left|t\big(\phi_{m,\infty}\big(L'_m\big(\phi_{n,k}(b)\big)\big)\big) - t\big(\phi_{n,\infty}(b)\big)\right| < 3\varepsilon/8 + t\big(\phi_{m,\infty}(c_3)\big) < 3\varepsilon/8 + \varepsilon/8 < \varepsilon$$

for all $t \in T(C)$ if $m > k$. Thus

(6) $t(\phi_{n,\infty}(b)) = \lim_{k\to\infty}(\lim_{m\to\infty} t(\phi_{m,\infty}(L'_m(\phi_{n,k}(b)))))$ for all $b \in C_n$ and $\tau \in T(C)$.

It follows that $r^\sharp \circ r^{\sharp -1} = \mathrm{id}_{T(C)}$. Thus we have shown that $r^\sharp$ is an affine continuous surjective map with an affine continuous inverse $r^{\sharp -1}$. To see $r^\sharp$ is face-preserving, let $\tau \in T(A)$, $t_1, t_2 \in T(C)$ and $0 \leqslant a \leqslant 1$ for which

$$r^\sharp(\tau) = a t_1 + (1-a) t_2.$$

Let $\tau_1, \tau_2 \in T(A)$ such that $r^\sharp(\tau_i) = t_i$, $i = 1, 2$. Then, since $r^{\sharp -1}$ is the inverse of $r^\sharp$, we see that

$$\tau = a\tau_1 + (1-a)\tau_2.$$

Fix a projection $p \in A$ and $s \in S_{u'}(K_0(C))$. Here we will use the notation from 10.7 and 10.2. One obtains a sequence of projections $e_n \in C_n$ such that

$$\lim_{n\to\infty} \|p_n p p_n - e_n\| = 0$$

or equivalently

$$\lim_{n\to\infty} \left\|L_n'(p) - e_n\right\| = 0.$$

We have shown that, for each $t \in T(C)$, $\lim_{n\to\infty} t(\phi_{n,\infty}(L_n'(p))$ exists. So $\lim_{n\to\infty} t(\phi_{n,\infty}(e_n))$ exists. If $p \in M_K(A)$ for some integer $K > 0$, by replacing C by $M_K(C)$ and p_n by $\mathrm{diag}(p_n, \ldots, p_n)$, we also obtain a projection $e_n \in C_n$ such that

$$\lim_{n,\infty} t\bigl(\phi_{n,\infty}\bigl(L_n'(p)\bigr)\bigr) = \lim_{n,\infty} t\bigl(\phi_{n,\infty}(e_n)\bigr). \tag{e7}$$

Since C is an inductive limit of C^*-algebras in $\mathcal{I}$, there exists $\sigma_s \in T(C)$ such that $s([e]) = \sigma_s(e)$ for any projection $e \in M_K(C)$ and for any integer $K \geqslant 1$ (recall that we use σ_s for $\sigma_s \otimes Tr$). Suppose that $\sigma_s, \tau_s \in T(C)$ such that $\sigma_s(e) = \tau_s(e)$ for all projections $e \in M_k(C)$ (for all integer $K \geqslant 1$). For any projection $p \in M_K(C)$, let e_n be a projection in C_n for which (e7) holds. Then

$$\lim_{n\to\infty} \sigma_s\bigl(\phi_{n,\infty} \circ L_n'(p)\bigr) = \lim_{n\to\infty} \sigma_s\bigl(\phi_{n,\infty}(e_n)\bigr) = \lim_{n\to\infty} \tau_s\bigl(\phi_{n,\infty}(e_n)\bigr) = \lim_{n\to\infty} \tau_s\bigl(\phi_{n,\infty} \circ L_n'(p)\bigr).$$

It follows that the map

$$r_\sharp(s)\bigl([p]\bigr) = r^{\sharp-1}(\sigma_s)(p) = \lim_{n\to\infty} \sigma_s\bigl(\phi_{n,\infty}\bigl(L_n'(p)\bigr)\bigr) = \lim_{n\to\infty} s\bigl(\bigl[\phi_{n,\infty}(e_n)\bigr]\bigr) \tag{e8}$$

is independent of the choices of τ_s and is well defined from $S_{u'}(K_0(C))$ to $S_u(K_0(A))$. (Here we extend L_n' and $\phi_{n,\infty}$ to $M_K(A)$ and $M_K(C)$ in the obvious way.) It is clear that $r_\sharp$ is affine.

Let $t \in S_u(K_0(A))$. Since A is a simple C^*-algebra with $TR(A) \leqslant 1$ (by 10.2), there exists $\tau_t \in T(A)$ such that τ_t induces t. Suppose that $\sigma_t \in T(A)$ such that $\tau_t(p) = \sigma_t(p)$ for all projections $p \in M_K(A)$ (for all integer $K \geqslant 1$). Let $e \in M_K(C_n)$ be a projection. Then (note that $C_k \subset A$)

$$\lim_{k\to\infty} \tau_t\bigl(\phi_{n,k}(e)\bigr) = \lim_{k\to\infty} \sigma_t\bigl(\phi_{n,k}(e)\bigr).$$

It follows that the map

$$r_\sharp'(t)\bigl(\bigl[\phi_{n,\infty}(e)\bigr]\bigr) = r^\sharp(t)\bigl(\phi_{n,\infty}(e)\bigr) = \lim_{k\to\infty} \tau_t\bigl(\phi_{n,k}(e)\bigr) = \lim_{k\to\infty} t\bigl(\bigl[\phi_{n,k}(e)\bigr]\bigr)$$

is independent of the choice of τ_t and it is well-defined affine map (where we view C_n as a C^*-subalgebra of A).

Now let $p \in A$ be a projection and $t \in S_u(K_0(A))$. By 10.2, t is induced by a trace $\tau_t \in T(A)$. One has, by (5) and (e8),

$$r_\sharp\big(r'_\sharp(t)\big)\big([p]\big) = \lim_{n\to\infty} r'_\sharp(t)\big(\big[\phi_{n,\infty}\big(L'_n(p)\big)\big]\big) = \lim_{n\to\infty}\Big(\lim_{k\to\infty}\tau_t\big(\phi_{n,k}\big(L'_n(p)\big)\big)\Big) = \tau_t(p) = t\big([p]\big).$$

It follows that $r_\sharp \circ r'_\sharp = \mathrm{id}_{S_{\mathrm{u}}(K_0(A))}$. On the other hand, let $e \in M_K(C_n)$ be a projection and $s \in S_{\mathrm{u}}(K_0(C))$. Let $\sigma_s \in T(C)$ which induces s. Then, by (6),

$$\begin{aligned} r'_\sharp\big(r_\sharp(s)\big)\big(\big[\phi_{n,\infty}(e)\big]\big) &= \lim_{k\to\infty} r_\sharp(s)\big(\big[\phi_{n,k}(e)\big]\big) = \lim_{k\to\infty}\Big(\lim_{m\to\infty}\sigma_s\big(\phi_{m,\infty}\big(L'_m\big(\phi_{n,k}(e)\big)\big)\big)\Big) \\ &= \sigma_s\big(\phi_{n,\infty}(e)\big) = s\big(\big[\phi_{n,\infty}(e)\big]\big). \end{aligned}$$

Thus $r'_\sharp \circ r_\sharp = \mathrm{id}_{S_{\mathrm{u}}(K_0(C))}$. □

10.9. Lemma. *Let A be a unital separable simple nuclear C^*-algebra with $TR(A) \leqslant 1$. Then the map $\lambda : T(A) \to S_{\mathrm{u}}(K_0(A))$ maps $\partial_{\mathrm{e}}(T(A))$ onto $\partial_{\mathrm{e}}(S_{\mathrm{u}}(K_0(A)))$. Moreover, if A is infinite-dimensional, $K_0(A)/tor(K_0(A)) \ncong \mathbb{Z}$. In particular, there is a unital simple AH-algebra B with no dimension growth described in* 10.1 *such that*

$$\big(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A)\big) = \big(K_0(B), K_0(B)_+, [1_B], K_1(B), T(B)\big).$$

Proof. We will apply Lemma 10.8. Let C be the inductive limit of C^*-algebras in $\mathcal{I}$ as described in 10.8. By [56, 1.11], the map from $T(\tilde{C})$ to $S_{\mathrm{u}}(K_0(\tilde{C}))$ maps extremal points onto extremal points. Let $t_0 \in T(\tilde{C})$ be the trace such that $t_0(c) = 0$ for all $c \in C$ and let $s_0 \in S_{\mathrm{u}}(K_0(\tilde{C}))$ such that $s_0(x) = 0$ for all $x \in j_*(K_0(C))$, where $j : C \to \tilde{C}$ is the embedding. Note that $T(\tilde{C})$ is the set of convex combinations of $\tau \in T(C)$ and t_0 and $S_{\mathrm{u}}(K_0(\tilde{C}))$ is the set of convex combinations of $s \in S_{\mathrm{u}'}(K_0(C))$ and s_0. Suppose that $\tau \in \partial_{\mathrm{e}}(T(A))$. Then, by 10.8, $r^\sharp(\tau) \in \partial_{\mathrm{e}}(T(C)) \subset \partial_{\mathrm{e}}(T(\tilde{C}))$. It follows that $r^\sharp(\tau)$ gives an extremal state s_τ in $S_{\mathrm{u}}(K_0(\tilde{C}))$. It follows that $s_\tau \in \partial_{\mathrm{e}}(S_{\mathrm{u}'}(K_0(C)))$. Note that $\lambda(\tau) = r'_\sharp(s_\tau)$. By 10.8, this shows that $\lambda(\partial_{\mathrm{e}}(T(A))) \subset \partial_{\mathrm{e}}(S_{\mathrm{u}}(K_0(A)))$. To see that $\lambda(\partial_{\mathrm{e}}(T(A))) = \partial_{\mathrm{e}}(S_{\mathrm{u}}(K_0(A)))$, let $s \in \partial_{\mathrm{e}}(S_{\mathrm{u}}(K_0(A)))$. Set

$$F = \big\{\tau \in T(A)\colon \lambda(\tau) = s\big\}.$$

It is clear that F is a closed and convex subset of $T(A)$. Furthermore it is a face. By the Krein–Milman theorem, it contains an extremal point t. Since F is a face, $t \in \partial_{\mathrm{e}}(T(A))$. Thus $\lambda(\partial_{\mathrm{e}}(T(A))) = \partial_{\mathrm{e}}(S_{\mathrm{u}}(K_0(A)))$.

To see $K_0(A)/tor(K_0(A)) \ncong \mathbb{Z}$ when A is infinite-dimensional, we note that A has (SP) by 3.2. Since A is simple, we obtain, for any integer $n > 0$, $n+1$ mutually orthogonal nonzero projections (see for example 5.5) $p_1, p_2, \ldots, p_n$ and q in A for which $1 = q + \sum_{i=1}^n p_i$, $[p_1] = [p_i]$ $(i = 1, 2, \ldots, n)$ and $[q] \leqslant [p_1]$. This implies that $K_0(A)/tor(K_0(A)) \ncong \mathbb{Z}$.

The last statement follows from 10.1 and the above. □

Now by 10.4 and 10.9 we have the following.

10.10. Theorem. *Let A and B be two unital separable nuclear simple C^*-algebras with $TR(A) \leqslant 1$ and $TR(B) \leqslant 1$ which satisfy the* AUCT. *Then $A \cong B$ if and only if*

$$\big(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A)\big) \cong \big(K_0(B), K_0(B)_+, [1_B], K_1(B), T(B)\big)$$

in the sense of 10.2.

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